VIRTUAL ELEMENT METHODS

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FEM on general meshes

Augmented FE are mentioned as early as Strang & Fix (1973)!

- **GFEM** [Babuška & Osborn (1983)]
- **Partition of Unity Method** [Babuška & Melenk (1997)]
- **XFEM** [eg. Fries & Belytschko (2010)]
- **PolygonalFEM** [Tabarraei & Sukumar (2004 & 2007)]
- **BEM-based FEM** [Copeland, Langer, & Pusch (2009)]
- Various dG approaches, eg:
  - **Polymorphic Nodal dG** [Gassner, Lörcher, Munz, & Hesthaven (2009)]
  - **Agglomeration dG** [Bassi, Botti, Colombo, & Rebay (2011)]
  - ...

**Driving idea:** generalize FE to polygons considering/adding particular shape functions that may help capturing the solution

**The VEM approach:** Generalize FE to polygons maintaining the ease of implementation of (polynomial) FE.
The Virtual Element Methods (VEM) principle for the generalization of FEM on polygons

- The VEM trial space contains, on each element, a space of polynomials, plus other functions.

- The degrees of freedom (dof) are carefully chosen in order to allow accurate dof-based computation when the polynomials are involved.

- It is shown that for the remaining part a result with the right order of magnitude and stability properties suffices.
Model problem and mesh partition

**Model problem:** On $\Omega \subset \mathbb{R}^2$, consider the symmetric variational problem

$$\begin{cases}
\text{find } u \in V := H^1_0(\Omega) \text{ such that } \\
a(u, v) = (f, v) \quad \forall v \in V,
\end{cases}$$

with $f \in L^2(\Omega)$ and

$$a(u, v) \leq M |u|_1 |v|_1, \quad a(v, v) \geq \alpha |v|^2_1 \quad \forall u, v \in V.$$

**Polygonal decompositions:** $\{T_h\}_h$ decompositions of $\Omega$ into non-overlapping polygons $K$

**Broken $H^1$-seminorm:** On $H^1(T_h) := \prod_{K \in T_h} H^1(K)$ define

$$|v|_{h,1} := \left( \sum_{K \in T_h} |\nabla v|^2_{0,K} \right)^{1/2}.$$
Abstract discrete problem

Given:

- A space $V_h \subset V$;
- A symmetric bilinear form $a_h : V_h \times V_h \rightarrow \mathbb{R}$ such that
  \[ a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} a^K_h(u_h, v_h) \quad \forall u_h, v_h \in V_h, \]
  where $a^K_h(\cdot, \cdot)$ is a bilinear form on $V_{h|K} \times V_{h|K}$;
- An element $f_h \in V'_h$.

We define the discrete problem

\[
\begin{cases}
\text{find } u_h \in V_h \text{ such that} \\
a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h
\end{cases}
\]
Assumptions on the discrete bilinear form

There exists an integer $k \geq 1$ such that for all $K \in \mathcal{T}_h$

- $\mathbb{P}_k(K) \subset V_h$.
- **$k$-consistency**: for all $p \in \mathbb{P}_k(K)$ and for all $v_h \in V_h|_K$,

  $$a^K_h(p, v_h) = a^K(p, v_h).$$

- **Stability**: there exist two positive constants $\alpha_*$ and $\alpha^*$, independent of $h$ and of $K$, such that

  $$\forall v_h \in V_h|_K \quad \alpha_* a^K(v_h, v_h) \leq a^K_h(v_h, v_h) \leq \alpha^* a^K(v_h, v_h).$$
Abstract convergence theorem

**Theorem**

Under all the above assumptions the discrete problem has a unique solution $u_h$ and

$$|u - u_h|_1 \lesssim \inf_{u_l \in V_h} |u - u_l|_1 + \inf_{u_\pi \in \oplus K \mathbb{P}_K(K)} |u - u_\pi|_{h,1} + \sup_{v \in V_h \setminus \{0\}} \frac{|(f, v) - \langle f_h, v \rangle|}{|v|_1}$$

$$\alpha_* |\delta_h|^2_1 \leq \langle f_h, \delta_h \rangle - \sum_K a^K_h(u_l, \delta_h) \quad (\delta_h := u_h - u_l)$$

$$= \langle f_h, \delta_h \rangle - \sum_K \left(a^K_h(u_l - u_\pi, \delta_h) + a^K_h(u_\pi, \delta_h)\right)$$

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$$= \langle f_h, \delta_h \rangle - (f, \delta_h) - \sum_K \left(a^K_h(u_l - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h)\right).$$
A Virtual Element Method

Consider the case \( a(u, v) = (\nabla u, \nabla v) \).

Fix \( K \in T_h \) with \( n \) edges, \( \mathbf{x}_K \) barycenter of \( K \), \( h_K \) diameter of \( K \), \( k \geq 1 \),

\[
\mathcal{B}_k(\partial K) := \{ v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \forall e \subset \partial K \};
\]

note that \( \dim(\mathcal{B}_k(\partial K)) = n + n(k - 1) = nk \). Then, define

\[
V^{K,k} = \{ v \in H^1(K) : v|_{\partial K} \in \mathcal{B}_k(\partial K), \Delta v|_K \in \mathbb{P}_{k-2}(K) \},
\]

with \( \mathbb{P}_{-1}(K) := \{0\} \). Note that

- \( \mathbb{P}_k(K) \subset V^{K,k} \);
- \( N^K := \dim V^{K,k} = nk + k(k - 1)/2 \).

Then, the global conforming VEM space is given by

\[
V_h := \{ v \in V : v|_K \in V^{K,k} \forall K \in T_h \}.
\]
VEM degrees of freedom

For instance, we may choose as dof for any $v_h \in V^{K,k}$,

- $V^{K,k}$ - The values of $v_h$ at the vertices.
- $E^{K,k}$ - For $k > 1$, $\forall e$ the values of $v_h$ at $k - 1$ distinct points
- $P^{K,k}$ - For $k > 1$, the moments

$$\frac{1}{|K|} \int_{K} m(x)v_h(x)\,dx \quad \forall m \in \left\{ \left( \frac{x - x_K}{h_K} \right)^s, \, |s| \leq k - 2 \right\}.$$
VEM vs FEM (recall that $N^K = nk + k(k - 1)/2$)

For $k = 1$ the two methods coincide!
VEM vs FEM (recall that $N^K = nk + k(k - 1)/2$)

Q.: What is VEM for $k = 1$?
VEM vs ??? (recall that $N^K = nk + k(k - 1)/2$)
Construction of *admissible* bilinear forms

Observe that, as for any \( v \in V^{K,k} \) and for any \( p \in \mathbb{P}_k(K) \),

\[
a^K(v, p) = \int_K \nabla v \cdot \nabla p \, dx = -\int_K v \Delta p \, dx + \int_{\partial K} v \frac{\partial p}{\partial n} \, ds,
\]

is exactly computable just using the local degrees of freedom.

**Example.** Take \( k = 2 \). Then,

\[
\int_K v \Delta p \, dx = |K| \Delta p \left( \frac{1}{|K|} \int_K v \, dx \right).
\]

Now, it is convenient to define the projection \( \Pi^K_k : V^{K,k} \rightarrow \mathbb{P}_k(K) \) as

\[
\left\{ \begin{array}{l}
a^K(\Pi^K_k v, q) = a^K(v, q) \quad \forall q \in \mathbb{P}_k(K) \\
\Pi^K_k v = \bar{v} := \frac{1}{n} \sum_{i=1}^{n} v(V_i) \quad (V_i = \text{vertices of } K)
\end{array} \right.
\]
Then,

\[ a^K_h(u, \nu) := a^K(\Pi^K_k u, \Pi^K_k \nu) + S^K(u - \Pi^K_k u, \nu - \Pi^K_k \nu) \quad \forall u, \nu \in V^{K,k}, \]

satisfies \textit{k-consistency} \textit{and stability} for every \( S^K(u, \nu) \) such that

\[ c_0 a^K(\nu, \nu) \leq S^K(\nu, \nu) \leq c_1 a^K(\nu, \nu) \quad \forall \nu \in V^{K,k} \quad \text{with } \Pi^K_k \nu = 0. \]

**Computing the RHS**

Similarly, define \( f_h := P^K_{k-2} f \) as the \( L^2(K) \)–projection of \( f \) onto \( \mathbb{P}_{k-2}(K) \)

Then, the associated right-hand side

\[ \langle f_h, \nu_h \rangle_K = \int_K f_h \nu_h \, dx = \int_K (P^K_{k-2} f) \nu_h \, dx = \int_K f \left( P^K_{k-2} \nu_h \right) \, dx \]

is exactly computable using the internal moments.
VEM on quadrilaterals: the case $k = 1$

Recall that if $\nu \in V^{K,1}$ then

$$\Delta \nu = 0 \quad \text{in } K, \quad \nu \in \mathbb{P}_1(e) \quad \forall e \subset \partial K, \quad \nu|_{\partial K} \in C^0(\partial K)$$

On parallelograms, $V^{K,1} = \mathbb{Q}^1(K)$.

And

$$a^K_h(u, \nu) := a^K(\Pi^K_1 u, \Pi^K_1 \nu) + S^K(u - \Pi^K_1 u, \nu - \Pi^K_1 \nu)$$

Thus, on parallelograms, $\text{VEM} \equiv \text{Bilinear FEM plus quadrature}$
An alternative point of view

 Actually, we may as well define: $V^{K,1} := \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}$ with

- $\chi_i|_e \in \mathbb{P}_1(e) \quad \forall e \subset \partial K$;
- $\chi_i(x_j) = \delta_{ij}$;
- $\mathbb{P}_1(K) \subset \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}$. 
An alternative point of view

Actually, we may as well define: \( V^{K,1} := \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\} \) with

- \( \chi_i|_e \in \mathbb{P}_1(e) \quad \forall e \subset \partial K; \)
- \( \chi_i(x_j) = \delta_{ij}; \)
- \( \mathbb{P}_1(K) \subset \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}. \)

And, with respect to this basis, \( a^K_h \) is given by

\[
a^K_h(\chi_i, \chi_j) = \int_K \nabla \chi_i \cdot \nabla \chi_j \, dx = \frac{1}{K} \mathbf{d}_i \cdot \mathbf{d}_j + (-1)^{i+j} \frac{\hat{T}_i \hat{T}_j}{K^2} \tau_K
\]

where

\[
\tau_K = \int_K |\nabla (\chi_1 + \chi_3)|^2 \, dx
\]
Square mesh: the stencil \((K = [0, 1]^2)\)

\[
\frac{\tau}{4} - \frac{1}{2} \quad -\frac{\tau}{2} \quad \frac{\tau}{4} - \frac{1}{2}
\]

\[
-\frac{\tau}{2} \quad \tau + 2 \quad -\frac{\tau}{2}
\]

\[
\frac{\tau}{4} - \frac{1}{2} \quad -\frac{\tau}{2} \quad \frac{\tau}{4} - \frac{1}{2}
\]

\[
\tau = \int_{[0,1]^2} |\nabla(\chi_1 + \chi_3)|^2 \, dx
\]
The values $\tau = 2/3$ (dashed line) and $\tau = 2$ (dotted line) are shown.
Examples

\[ \begin{array}{ccc}
\tau = 2 \\
0 & -1 & 0 \\
-1 & +4 & -1 \\
0 & -1 & 0 \\
\end{array} \quad \begin{array}{ccc}
\tau = 4/3 \\
-\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \\
-\frac{2}{3} & +\frac{10}{3} & -\frac{2}{3} \\
-\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \\
\end{array} \]

5-points Laplace

9-points Laplace

\[ \begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & +\frac{8}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\end{array} \quad \begin{array}{ccc}
-\frac{1}{2} & 0 & -\frac{1}{2} \\
0 & +2 & 0 \\
-\frac{1}{2} & 0 & -\frac{1}{2} \\
\end{array} \]

Exact \( \mathbb{Q}_1 \)

Hourglass

All these result from applying quadrature to the \( \mathbb{Q}_1 \text{-FE} \) and from the VEM bilinear form using different spaces.
References

- C., Manzini, & Russo *Nine-point discrete Laplace operators for general quadrilateral meshes*. Submitted to CMAME.


**WORKSHOP** Disretization Methods for Polygonal and Polyhedral Meshes

Milan, 17-19 September 2012

http://k.matapp.unimib.it/WSVEM-2012