VIRTUAL ELEMENT METHODS

Andrea Cangiani

UNIVERSITY OF LEICESTER

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FEM on general meshes

Augmented FE are mentioned as early as Strang & Fix (1973)!

- GFEM [Babuška & Osborn (1983)]
- Partition of Unity Method [Babuška & Melenk (1997)]
- XFEM [*eg.* Fries & Belytschko (2010)]
- PolygonalFEM [Tabarraei & Sukumar (2004 & 2007)]
- BEM-based FEM [Copeland, Langer, & Pusch (2009)]
- Various dG approaches, *eg.*
    - Polymorphic Nodal dG [Gassner, Lörcher, Munz, & Hesthaven (2009)]
    - Agglomeration dG [Bassi, Botti, Colombo, & Rebay (2011)]
- ... 

Driving idea: generalize FE to polygons considering/adding particular shape functions that may help capturing the solution

The VEM approach: Generalize FE to polygons maintaining the ease of implementation of (polynomial) FE.

The Virtual Element Methods (VEM) principle

for the generalization of FEM on polygons

- The VEM trial space contains, on each element, a space of polynomials, plus other functions.
- The degrees of freedom (dof) are carefully chosen in order to allow *accurate* dof-based computation when the polynomials are involved.
- It is shown that for the remaining part a result with the *right order of magnitude and stability* properties suffices.

Model problem and mesh partition

Model problem: On $\Omega \subset \mathbb{R}^2$, consider the symmetric variational problem

$$\begin{cases} \text{find } u \in V := H_0^1(\Omega) \text{ such that} \\ a(u,v) = (f,v) \quad \forall v \in V, \end{cases}$$

with $f \in L^2(\Omega)$ and

$$a(u,v) \leq M\,|u|_1|v|_1, \qquad a(v,v) \geq \alpha\,|v|_1^2 \qquad \forall u, v \in V.$$

Polygonal decompositions: $\{\mathcal{T}_h\}_h$ decompositions of Ω into non-overlapping polygons K

Broken H^1-seminorm: On $H^1(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^1(K)$ define

$$|v|_{h,1} := \Big(\sum_{K \in \mathcal{T}_h} |\nabla v|_{0,K}^2 \Big)^{1/2}.$$

Abtract discrete problem

Given:

- A space $V_h \subset V$;
- A symmetric bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ such that

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} a_h^K(u_h, v_h) \quad \forall\, u_h,\, v_h \, \in V_h,$$

  where $a_h^K(\cdot,\cdot)$ is a bilinear form on $V_{h|K} \times V_{h|K}$;
- An element $f_h \in V_h'$.

We define the discrete problem

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) = \langle f_h, v_h \rangle \qquad \forall\, v_h \in V_h \end{cases}$$

Assumptions on the discrete bilinear form

There exists an integer $k \geq 1$ such that for all $K \in \mathcal{T}_h$

- $\mathbb{P}_k(K) \subset V_h$.
- k-consistency: for all $p \in \mathbb{P}_k(K)$ and for all $v_h \in V_{h|K}$,

$$a_h^K(p, v_h) = a^K(p, v_h).$$

- Stability: there exist two positive constants α_* and α^*, independent of h and of K, such that

$$\forall v_h \in V_{h|K} \qquad \alpha_* \, a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^* \, a^K(v_h, v_h).$$

Abstract convergence theorem

Theorem

Under all the above assumptions the discrete problem has a unique solution u_h and

$$|u - u_h|_1 \lesssim \inf_{u_I \in V_h} |u - u_I|_1 + \inf_{u_\pi \in \oplus_K \mathbb{P}_k(K)} |u - u_\pi|_{h,1} + \sup_{v \in V_h \setminus \{0\}} \frac{|(f, v) - \langle f_h, v\rangle|}{|v|_1}$$

$$\begin{aligned}
\alpha_* |\delta_h|_1^2 &\leq \langle f_h, \delta_h\rangle - \textstyle\sum_K a_h^K(u_I, \delta_h) \qquad (\delta_h := u_h - u_I) \\
&= \langle f_h, \delta_h\rangle - \textstyle\sum_K \left(a_h^K(u_I - u_\pi, \delta_h) + a_h^K(u_\pi, \delta_h)\right) \\
&= \langle f_h, \delta_h\rangle - \textstyle\sum_K \left(a_h^K(u_I - u_\pi, \delta_h) + a^K(u_\pi, \delta_h)\right) \\
&= \langle f_h, \delta_h\rangle - (f, \delta_h) - \textstyle\sum_K \left(a_h^K(u_I - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h)\right).
\end{aligned}$$

A Virtual Element Method

Consider the case $a(u, v) = (\nabla u, \nabla v)$.

Fix $K \in \mathcal{T}_h$ with n edges, $\mathbf{x}_K$ barycenter of K, h_K diameter of K, $k \geq 1$,

$$\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) : \ v_{|e} \in \mathbb{P}_k(e) \ \forall e \subset \partial K\};$$

note that $\dim(\mathbb{B}_k(\partial K)) = n + n(k-1) = nk$. Then, define

$$V^{K,k} = \{v \in H^1(K) : \ v_{|\partial K} \in \mathbb{B}_k(\partial K), \ \Delta v_{|K} \in \mathbb{P}_{k-2}(K)\},$$

with $\mathbb{P}_{-1}(K) := \{0\}$. Note that

- $\mathbb{P}_k(K) \subset V^{K,k}$;
- $\mathrm{N}^K := \dim V^{K,k} = nk + k(k-1)/2$.

Then, the global conforming VEM space is given by

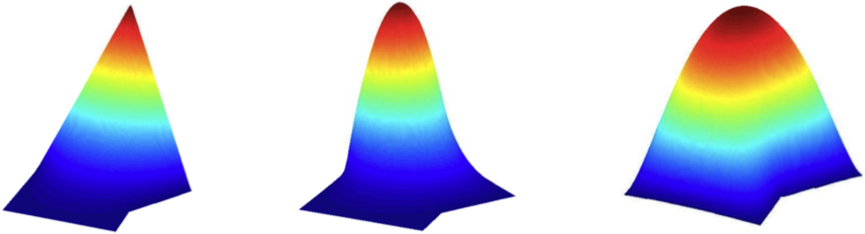
$$V_h := \{v \in V : \ v|_K \in V^{K,k} \ \forall K \in \mathcal{T}_h\}$$

VEM degrees of freedom

For instance, we may choose as dof for any $v_h \in V^{K,k}$,

- $\mathcal{V}^{K,k}$ - The values of v_h at the *vertices*.
- $\mathcal{E}^{K,k}$ - For $k > 1$, $\forall e$ the values of v_h at $k-1$ distinct points
- $\mathcal{P}^{K,k}$ - For $k > 1$, the moments

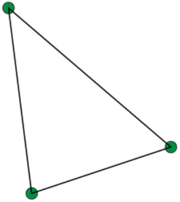
$$\frac{1}{|K|}\int_K m(\mathbf{x}) v_h(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad \forall m \in \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^{\mathbf{s}}, \ |\mathbf{s}| \leq k-2 \right\}.$$



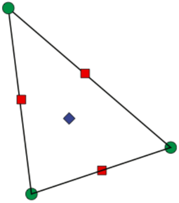
VEM vs FEM (recall that $N^K = nk + k(k-1)/2$)

VEM

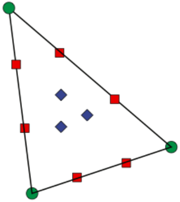
$k = 1$



$k = 2$

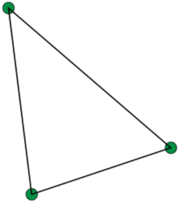


$k = 3$

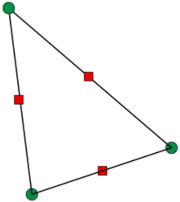


FEM

$k = 1$



$k = 2$



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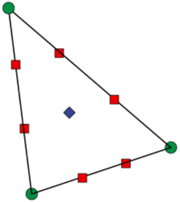
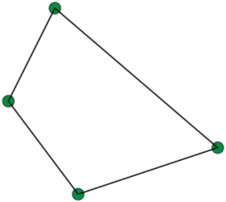


For $k = 1$ the two methods coincide!

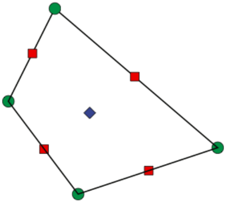
VEM vs FEM (recall that $N^K = nk + k(k-1)/2$)

VEM

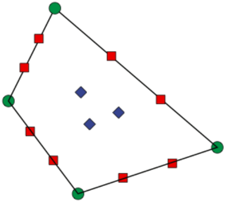
$k = 1$



$k = 2$

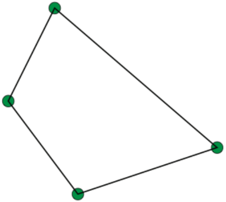


$k = 3$

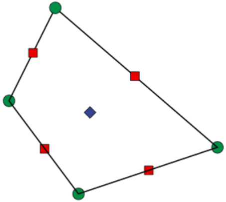


FEM

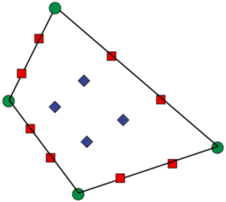
$k = 1$



$k = 2$



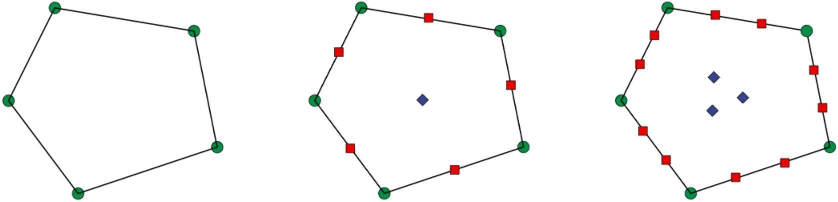
$k = 3$



Q.: What is VEM for $k = 1$?

VEM vs ??? (recall that $N^K = nk + k(k-1)/2$)

VEM $k = 1$ $k = 2$ $k = 3$



Construction of *admissible* bilinear forms

Observe that, as for any $v \in V^{K,k}$ and for any $p \in \mathbb{P}_k(K)$,

$$a^K(v,p) = \int_K \nabla v \cdot \nabla p \,\mathrm{d}x = -\int_K v\,\Delta p\,\mathrm{d}x + \int_{\partial K} v\,\frac{\partial p}{\partial n}\,\mathrm{d}s,$$

is exacly computable just using the local degrees of freedom.

Example. Take $k = 2$. Then,

$$\int_K v\Delta p\,dx = |K|\Delta p\left(\frac{1}{|K|}\int_K v\,dx\right).$$

Now, it is convenient to define the projection $\Pi_k^K : \ V^{K,k} \to \mathbb{P}_k(K)$ as

$$\begin{cases} a^K(\Pi_k^K v, q) = a^K(v,q) \quad \forall q \in \mathbb{P}_k(K) \\ \overline{\Pi_k^K v} = \overline{v} := \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} v(V_i) \qquad \big(V_i = \text{ vertices of } K\big) \end{cases}$$

Then,

$$a_h^K(u,v) := a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v) \quad \forall u, v \in V^{K,k},$$

satisfies k-consistency and stability for every $S^K(u,v)$ such that

$$c_0 a^K(v,v) \le S^K(v,v) \le c_1 a^K(v,v) \quad \forall v \in V^{K,k} \quad \text{with } \Pi_k^K v = 0.$$

Computing the RHS

Similarly, define $f_h := P_{k-2}^K f$ as the $L^2(K)-$projection of f onto $\mathbb{P}_{k-2}(K)$

Then, the associated right-hand side

$$\langle f_h, v_h \rangle_K = \int_K f_h\, v_h \,\mathrm{d}x = \int_K (P_{k-2}^K f)\, v_h \,\mathrm{d}x = \int_K f\,(P_{k-2}^K v_h)\,\mathrm{d}x$$

is exactly computable using the internal moments.

VEM on quadrilaterals: the case $k = 1$

Recall that if $v \in V^{K,1}$ then

$$\Delta v = 0 \quad \text{in } K, \qquad v \in \mathbb{P}_1(e) \quad \forall e \subset \partial K, \qquad v|_{\partial K} \in C^0(\partial K)$$

On parallelograms, $V^{K,1} = \mathbb{Q}^1(K)$.

And

$$a_h^K(u, v) := a^K(\Pi_1^K u, \Pi_1^K v) + S^K(u - \Pi_1^K u, v - \Pi_1^K v)$$

Thus, on parallelograms, VEM $\equiv$ Bilinear FEM plus quadrature

An alternative point of view

Actually, we may as well define: $V^{K,1} := \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}$ with

- $\chi_i|_e \in \mathbb{P}_1(e) \quad \forall e \subset \partial K$;
- $\chi_i(\mathbf{x}_j) = \delta_{ij}$;
- $\mathbb{P}_1(K) \subset \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}$.

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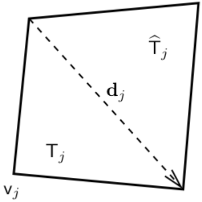
- $\chi_i(\mathbf{x}_j) = \delta_{ij}$;
- $\mathbb{P}_1(K) \subset \text{span}\{\chi_1, \chi_2, \chi_3, \chi_4\}$.

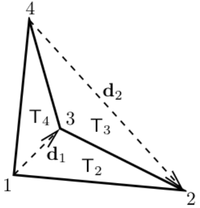
And, with respect to this basis, a_h^K is given by

$$a_h^K(\chi_i, \chi_j) = \int_K \nabla\chi_i \cdot \nabla\chi_j \, dx = \frac{1}{K}\mathbf{d}_i \cdot \mathbf{d}_j + (-1)^{i+j} \frac{\widehat{T}_i \widehat{T}_j}{K^2} \tau_K$$

where

$$\tau_K = \int_K |\nabla(\chi_1 + \chi_3)|^2 \, dx$$



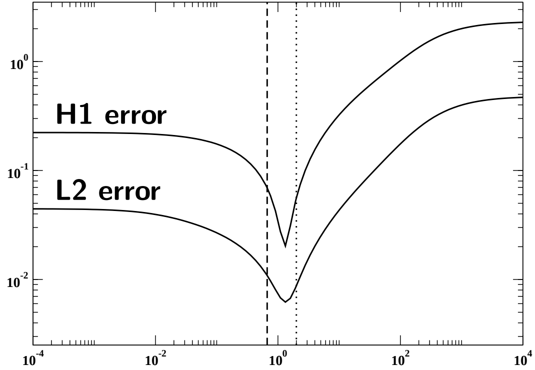


Square mesh: the stencil ($K = [0,1]^2$)

$$
\begin{array}{ccccc}
\frac{\tau}{4} - \frac{1}{2} & \text{——} & -\frac{\tau}{2} & \text{——} & \frac{\tau}{4} - \frac{1}{2} \\
| & & | & & | \\
-\frac{\tau}{2} & \text{——} & \tau + 2 & \text{——} & -\frac{\tau}{2} \\
| & & | & & | \\
\frac{\tau}{4} - \frac{1}{2} & \text{——} & -\frac{\tau}{2} & \text{——} & \frac{\tau}{4} - \frac{1}{2}
\end{array}
$$

$$\tau = \int_{[0,1]^2} |\nabla(\chi_1 + \chi_3)|^2 \, dx$$

Method robustness



The values $\tau = 2/3$ (dashed line) and $\tau = 2$ (dotted line) are shown.

Examples

5-points Laplace, $\tau = 2$

$$\begin{array}{ccccc} 0 & — & -1 & — & 0 \\ | & & | & & | \\ -1 & — & +4 & — & -1 \\ | & & | & & | \\ 0 & — & -1 & — & 0 \end{array}$$

9-points Laplace, $\tau = 4/3$

$$\begin{array}{ccccc} -\frac{1}{6} & — & -\frac{2}{3} & — & -\frac{1}{6} \\ | & & | & & | \\ -\frac{2}{3} & — & +\frac{10}{3} & — & -\frac{2}{3} \\ | & & | & & | \\ -\frac{1}{6} & — & -\frac{2}{3} & — & -\frac{1}{6} \end{array}$$

Exact $\mathbb{Q}_1$, $\tau = 2/3$

$$\begin{array}{ccccc} -\frac{1}{3} & — & -\frac{1}{3} & — & -\frac{1}{3} \\ | & & | & & | \\ -\frac{1}{3} & — & +\frac{8}{3} & — & -\frac{1}{3} \\ | & & | & & | \\ -\frac{1}{3} & — & -\frac{1}{3} & — & -\frac{1}{3} \end{array}$$

Hourglass, $\tau = 0$

$$\begin{array}{ccccc} -\frac{1}{2} & — & 0 & — & -\frac{1}{2} \\ | & & | & & | \\ 0 & — & +2 & — & 0 \\ | & & | & & | \\ -\frac{1}{2} & — & 0 & — & -\frac{1}{2} \end{array}$$

All these result from applying quadrature to the $\mathbb{Q}_1$–FE *and* from the VEM bilinear form using different spaces.

References

- C., Manzini, & Russo Nine-point discrete Laplace operators for general quadrilateral meshes. Submitted to CMAME.
- Beirão da Veiga, Brezzi, C., Manzini, Marini, & Russo Basic principles of Virtual Element Methods. Submitted to M^3AS. IMATI Tech. Rep., Pavia, 2012.
- Brezzi, C., Manzini, & Russo Mimetic Methods and Virtual Elements Methods for diffusion problems on polygonal meshes. IMATI Tech. Rep., Pavia, 2012.

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