A Nitsche-based domain decomposition method for hypersingular integral equations

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in collaboration with

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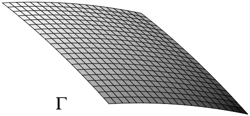
Model Problem

Find $u \in \tilde{H}^{1/2}(\Gamma)$:

$$Wu = f \quad \text{on } \Gamma$$



Hypersingular integral operator:

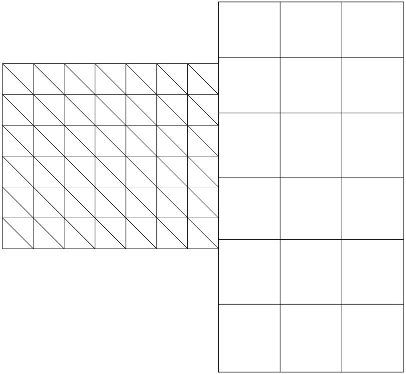
$$Wu(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma$$

Energy norm [Stephan '87]:

$$\|u\|^2_{1/2,\sim,\Gamma} = \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} = \inf_{U|_\Gamma = u}\|U\|^2_{H^1(\mathbb{R}^3)} + \int_\Gamma \frac{u^2(x)}{\operatorname{dist}(x,\partial\Gamma)}\,dS_x$$

$$= \|u\|^2_{1/2,\Gamma} \quad + \quad \text{essential boundary condition}$$

Domain decomposition

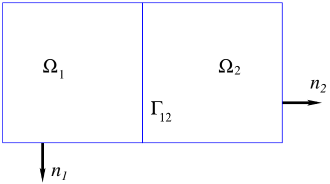


Coupling by

- alternating Schwarz method
- mortar method or Lagrangian multipliers
- Nitsche coupling or penalty method

Domain decomposition with Nitsche coupling: Laplacian

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma := \partial\Omega \end{cases}$$



$$u_i := u|_{\Omega_i}, \quad [u] := (u_2 - u_1)|_{\Gamma_{12}} = 0$$

$$T_i u := \frac{\partial u_i}{\partial n_i}, \qquad T_1 u = -T_2 u \quad \text{on } \Gamma_{12}$$

$$\langle \nabla u, \nabla v \rangle_{\Omega_i} - \langle \frac{\partial u_i}{\partial n_i}, v \rangle_{\Gamma_{12}} = \langle f, v \rangle_{\Omega_i} \quad \forall v \in H^1(\Omega_i), \; v|_\Gamma = 0$$

$$\sum_i \langle \nabla u, \nabla v_i \rangle_{\Omega_i} - \sum_i \langle T_i u, v_i \rangle_{\Gamma_{12}} = \langle f, v \rangle_\Omega$$

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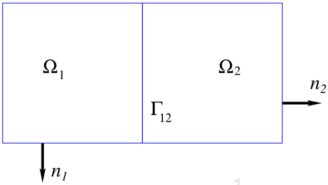

$$(T_1 u + T_2 u) - \frac{2\nu}{h}[u] = 0$$

$$(T_1 u + T_2 u) + \frac{2\nu}{h}[u] = 0, \quad \nu > 0$$

$$\begin{aligned}
\sum_i \langle T_i u, v_i\rangle_{\Gamma_{12}} &= \langle \frac{1}{2} T_1 u_1 + \frac{1}{2} T_1 u_1, v_1\rangle_{\Gamma_{12}} + \langle \frac{1}{2} T_2 u_2 + \frac{1}{2} T_2 u_2, v_2\rangle_{\Gamma_{12}} \\
&= \langle \frac{1}{2} T_1 u_1 + \frac{1}{2}(-T_2 u_1 + \frac{2\nu}{h}[u]), v_1\rangle_{\Gamma_{12}} \\
&\quad + \langle \frac{1}{2} T_2 u_2 + \frac{1}{2}(-T_1 u_2 - \frac{2\nu}{h}[u]), v_2\rangle_{\Gamma_{12}} \\
&= -\frac{1}{2}\langle T_1 u_1 - T_2 u_2, [v]\rangle_{\Gamma_{12}} + \frac{\nu}{h}\langle [u], [v]\rangle_{\Gamma_{12}}.
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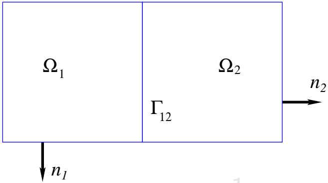

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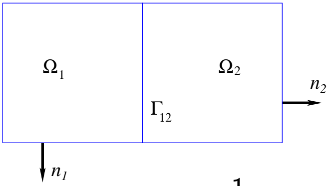

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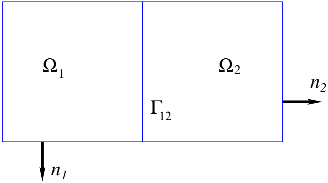

$$\langle \nabla_{\mathcal{T}} u, \nabla_{\mathcal{T}} v \rangle_{\Omega} + \frac{1}{2} \langle T_1 u_1 - T_2 u_2, [v] \rangle_{\Gamma_{12}} + \frac{\nu}{h} \langle [u], [v] \rangle_{\Gamma_{12}} = \langle f, v \rangle_{\Omega} \qquad \forall v \ldots, \ v|_{\Gamma} = 0$$

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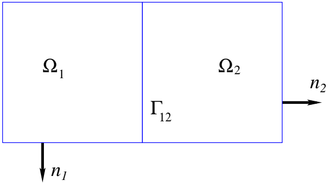

$$\langle \nabla_{\mathcal{T}} u, \nabla_{\mathcal{T}} v \rangle_\Omega + \frac{1}{2}\langle T_1 u_1 - T_2 u_2, [v] \rangle_{\Gamma_{12}}$$

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$$(\sigma = \pm 1, \quad \nu > 0)$$

$$a_\sigma(u, v) = \langle f, v \rangle_\Omega$$

$$\forall v = (v_1, v_2) \in H^1(\Omega_1) \times H^1(\Omega_2), \quad v_1 = v_2 = 0 \quad \text{on } \Gamma$$

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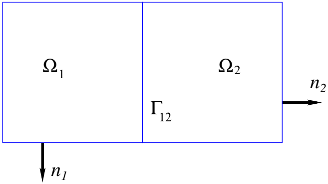

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$\sigma = 1$: symmetric $\qquad$ $\sigma = -1$: block-antisymmetric

a_{-1} is elliptic:

$$a_{-1}(v,v) = \langle \nabla_{\mathcal{T}} v, \nabla_{\mathcal{T}} v\rangle_\Omega + \frac{\nu}{h}\langle [v],[v]\rangle_{\Gamma_{12}} = |v|^2_{H^1(\Omega_1\times\Omega_2)} + \frac{\nu}{h}\|[v]\|^2_{0,\Gamma_{12}}$$

Finite elements with Nitsche coupling:

standard approximation spaces over subdomains
without interface condition

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Remaining topics

- Integration by parts for the hypersingular operator
- Domain decomposition with Nitsche coupling
- Numerical results
- Conclusions, References

Integration by parts for the hypersingular operator

$$Wu = \mathrm{curl}_\Gamma\, V\, \mathbf{curl}_\Gamma\, u \qquad \text{[Maue '49, Nédélec '82]}$$

with $\quad V\psi(x) := \dfrac{1}{4\pi}\displaystyle\int_\Gamma \psi(y)\frac{1}{|x-y|}\,dS_y \quad$ (single layer operator)

Integration by parts yields well-known formula

$$\langle Wu, v\rangle_\Gamma = \langle V\,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_\Gamma\, v\rangle_\Gamma \quad \forall u, v \in \tilde{H}^{1/2}(\Gamma),$$

and for smooth v without homogeneous boundary condition:

$$\langle Wu, v\rangle_\Gamma = \langle V\,\mathbf{curl}_\Gamma\, u, \mathbf{curl}_\Gamma\, v\rangle_\Gamma + \langle \mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\, u, v\rangle_{\partial\Gamma}$$

- $v \in H^{1/2}(\Gamma)$ has no well-defined trace on $\partial\Gamma$
- formula is not well defined for the continuous setting

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Lemma [Gatica, H., Healey '09]

$$Wu = f \in (H^{1/2}(\Gamma))' \qquad \Longrightarrow \qquad \mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\, u \in H^{-1/2}(\partial\Gamma)$$

In particular, formula applies to $v \in H^1(\Gamma)$ (boundary elements).

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- Numerical results
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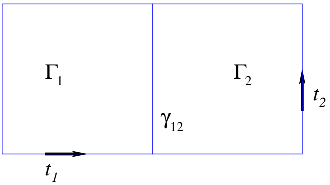
Domain decomposition with Nitsche coupling

$$\text{solve} \qquad Wu = f \qquad \text{on } \Gamma$$

$$\text{int.b.p.} \quad \langle Wu, v_i\rangle_{\Gamma_i} = \langle V\, \mathbf{curl}_\Gamma\, u, \mathbf{curl}_{\Gamma_i}\, v_i\rangle_{\Gamma_i} + \langle \mathbf{t}\cdot V\, \mathbf{curl}_\Gamma\, u, v_i\rangle_{\partial\Gamma_i}$$



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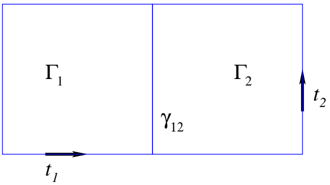

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Domain decomposition with Nitsche coupling

bilinear form: $a_\sigma(v,w) := \langle V\,\mathbf{curl}_\mathcal{T}\, v, \mathbf{curl}_\mathcal{T}\, w\rangle_\Gamma + \nu\langle [v],[w]\rangle_\gamma$

$$+ \frac{1}{2}\langle T_1 v - T_2 v, [w]\rangle_\gamma + \frac{\sigma}{2}\langle [v], T_1 w - T_2 w\rangle_\gamma$$

norm: $\|v\|^2_{s,\mathcal{T},\nu} := |v|^2_{s,\mathcal{T}} + \nu\|[v]\|^2_{0,\gamma}$

discretization: quasi-uniform meshes on Γ_i

$H_{i,h} \subset H^1(\Gamma_i)$ with zero trace on $\partial\Gamma_i \cap \partial\Gamma$, $H_h := \Pi_i H_{i,h}$

Boundary element method with Nitsche domain decomposition:

$$u_h \in H_h : \qquad a_\sigma(u_h, v) = \langle f, v\rangle_\Gamma \quad \forall v \in H_h$$

Domain decomposition with Nitsche coupling

bilinear form: $a_\sigma(v,w) := \langle V\,\mathbf{curl}_{\mathcal{T}}\, v, \mathbf{curl}_{\mathcal{T}}\, w\rangle_\Gamma + \nu\langle [v],[w]\rangle_\gamma$
$$+\frac{1}{2}\langle T_1 v - T_2 v, [w]\rangle_\gamma + \frac{\sigma}{2}\langle [v], T_1 w - T_2 w\rangle_\gamma$$

norm: $\|v\|^2_{s,\mathcal{T},\nu} := |v|^2_{s,\mathcal{T}} + \nu\|[v]\|^2_{0,\gamma}$

discretization: quasi-uniform meshes on Γ_i

$H_{i,h} \subset H^1(\Gamma_i)$ with zero trace on $\partial\Gamma_i \cap \partial\Gamma$, $H_h := \Pi_i H_{i,h}$

Boundary element method with Nitsche domain decomposition:

$$u_h \in H_h : \qquad a_\sigma(u_h, v) = \langle f, v\rangle_\Gamma \quad \forall v \in H_h$$

Domain decomposition with Nitsche coupling

Disadvantages:

- additional non-local operator
- symmetric case not uniformly elliptic

Advantages:

- no additional coupling variable (e.g. Lagrangian multiplier)
- symmetry possible
- elliptic system
- generic coupling for various discretizations

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Domain decomposition with Nitsche coupling

Discrete ellipticity, antisymmetric case:

$$
\begin{aligned}
a_{-1}(v,v) &= \langle V \,\mathbf{curl}_{\mathcal{T}}\, v, \mathbf{curl}_{\mathcal{T}}\, v\rangle_\Gamma + \nu\langle [v],[v]\rangle_\gamma \\
&\gtrsim \|\,\mathbf{curl}_{\mathcal{T}}\, v\|^2_{-1/2,\sim,\Gamma} + \nu\|[v]\|^2_{0,\gamma} \\
&\gtrsim \quad |v|^2_{1/2,\mathcal{T}} \qquad\quad + \nu\|[v]\|^2_{0,\gamma} \quad = \|v\|^2_{1/2,\mathcal{T},\nu} \qquad \forall v \in H_h
\end{aligned}
$$

Domain decomposition with Nitsche coupling

Discrete ellipticity, symmetric case:

$$a_1(v,v) = \langle V\,\mathbf{curl}_{\mathcal{T}}\, v, \mathbf{curl}_{\mathcal{T}}\, v\rangle_\Gamma + \nu\langle [v],[v]\rangle_\gamma + \langle T_1 v - T_2 v, [v]\rangle_\gamma$$

$$\langle T_1 v - T_2 v, [v]\rangle_\gamma \le \|T_1 v - T_2 v\|_{0,\gamma}\|[v]\|_{0,\gamma} \lesssim \delta\|T_1 v - T_2 v\|^2_{0,\gamma} + \frac{1}{\delta}\|[v]\|^2_{0,\gamma}$$

$$\|T_i v\|^2_{0,\gamma} = \|\mathbf{t}_i \cdot V\,\mathbf{curl}_\Gamma\, v\|^2_{0,\gamma}$$

$\lesssim \epsilon^{-2}\|V\,\mathbf{curl}_\Gamma\, v\|^2_{1/2+\epsilon,\Gamma}$	trace theorem
$\lesssim \epsilon^{-2}\|\mathbf{curl}_\Gamma\, v\|^2_{-1/2+\epsilon,\sim,\mathcal{T}}$	cont. of V, norm decomposition
$\lesssim \epsilon^{-2}\epsilon^{-1}\|\mathbf{curl}_\Gamma\, v\|^2_{-1/2+\epsilon,\mathcal{T}}$	equivalence of norms
$\lesssim \epsilon^{-3}\lvert v\rvert^2_{1/2+\epsilon,\mathcal{T}}$	cont. of $\mathbf{curl}_\Gamma$, quotient space arg.
$\lesssim \epsilon^{-3}h^{-2\epsilon}\lvert v\rvert^2_{1/2+0,\mathcal{T}}$	inverse property, $v \in H_h$

$$a_1(v,v) \gtrsim \left(1 - \frac{\delta}{\epsilon^3}h^{-2\epsilon}\right)|v|^2_{1/2,\mathcal{T}} + \left(\nu - \frac{1}{\delta}\right)\|[v]\|^2_{0,\gamma}$$

Domain decomposition with Nitsche coupling

Discrete ellipticity, symmetric case:

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$$\begin{aligned}\| T_i v\|_{0,\gamma}^2 &= \|\mathbf{t}_i \cdot V\,\mathbf{curl}_\Gamma\, v\|_{0,\gamma}^2 \\ &\lesssim \epsilon^{-2}\| V\,\mathbf{curl}_\Gamma\, v\|_{1/2+\epsilon,\Gamma}^2 && \text{trace theorem} \\ &\lesssim \epsilon^{-2}\|\mathbf{curl}_\Gamma\, v\|_{-1/2+\epsilon,\sim,\mathcal{T}}^2 && \text{cont. of } V\text{, norm decomposition} \\ &\lesssim \epsilon^{-2}\epsilon^{-1}\|\mathbf{curl}_\Gamma\, v\|_{-1/2+\epsilon,\mathcal{T}}^2 && \text{equivalence of norms} \\ &\lesssim \epsilon^{-3}|v|_{1/2+\epsilon,\mathcal{T}}^2 && \text{cont. of } \mathbf{curl}_\Gamma\text{, quotient space arg.} \\ &\lesssim \epsilon^{-3}h^{-2\epsilon}|v|_{1/2+0,\mathcal{T}}^2 && \text{inverse property,} \quad v \in H_h\end{aligned}$$

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Conclusion:

$$\epsilon := |\log h|^{-1}, \quad \delta \sim |\log h|^{-3}, \quad \nu > O(|\log h|^3)$$

generates discrete ellipticity

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Domain decomposition with Nitsche coupling

Theorem (a priori error estimate) [F.C., N. Heuer]
$\nu > 0$ (antisymmetric case), $\nu > O(|\log h|^3)$ (symmetric case)

$$\|u - u_h\|_{1/2,\mathcal{T},\nu} \lesssim \max\{\nu^{-1/2}|\log h|^{3/2}, \nu^{1/2}|\log h|^{1/2}\}h^{r-1/2}\|u\|_{r,\Gamma}$$

$$(r < 1)$$

- antisymmetric case : $\sigma = -1$, $\nu = O(|\log h|)$:

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Proof.
- *Step 1.* For $v \in H_h$:

$$\|u - u_h\|_{1/2,\mathcal{T},\nu} \leq \|u - v\|_{1/2,\mathcal{T},\nu} + \|u_h - v\|_{1/2,\mathcal{T},\nu}$$
$$\lesssim \|u - v\|_{1/2,\mathcal{T},\nu} + \sup_{w \in H_h \setminus \{0\}} \frac{a_\sigma(u_h - v, w)}{\|w\|_{1/2,\mathcal{T},\nu}}$$

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Proof.
- *Step 2.* Deal with :

$$\begin{aligned}\frac{a_\sigma(u_h - v, w)}{\|w\|_{1/2,\mathcal{T},\nu}} &= \frac{a_\sigma(u - v, w)}{\|w\|_{1/2,\mathcal{T},\nu}} \\ &\lesssim (\epsilon^{-1} + \nu^{-1/2}\epsilon^{-3/2} + h^{-\epsilon}\epsilon^{-1} + \nu^{1/2}\epsilon^{-1/2}) \\ &\quad \underbrace{\|u - v\|_{1/2+\epsilon,\mathcal{T}}}_{\lesssim h^{r-\epsilon-1/2}\|u\|_{r,\Gamma}}\end{aligned}$$

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$$(r < 1)$$

Proof.
- *Step 3.* Combine Steps 1 & 2 :

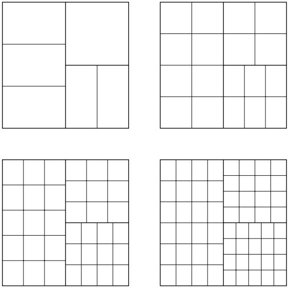
$$\|u - u_h\|_{1/2,\mathcal{T},\nu} \lesssim (\nu^{-1/2}\epsilon^{-3/2}h^{-\epsilon} + \epsilon^{-1}h^{-2\epsilon} + \nu^{1/2}\epsilon^{-1/2}h^{-\epsilon}) h^{r-1/2}\|u\|_{r,\Gamma}$$

- *Step 4.* Conclude with : $\epsilon = |\log h|^{-1}$ and

$$|\log h| = (\nu^{-1/4}|\log h|^{3/4})(\nu^{1/4}|\log h|^{1/4}) \leq \frac{1}{2}(\nu^{-1/2}|\log h|^{3/2} + \nu^{1/2}|\log h|^{1/2})$$

- Integration by parts for the hypersingular operator
- Domain decomposition with Nitsche coupling
- Numerical results
- Conclusions, References

Numerical results



$$Wu = 1 \qquad \text{on} \quad \Gamma = (0,1)^2 \times \{0\}$$

H_h : continuous p/w bilinear on uniform meshes

$$\langle V\,\mathbf{curl}_{\mathcal{T}}\,u, \mathbf{curl}_{\mathcal{T}}\,v\rangle_{\mathcal{T}} + \frac{1}{2}\sum_{0\le i<j}\langle T_i u - T_j u, [v]\rangle_\gamma + \nu \sum_{0\le i<j}\langle [u],[v]\rangle_\gamma$$

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Error calculation:

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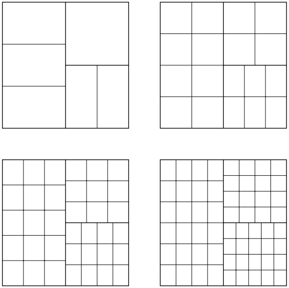

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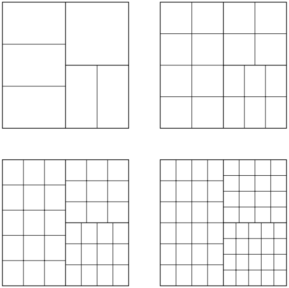

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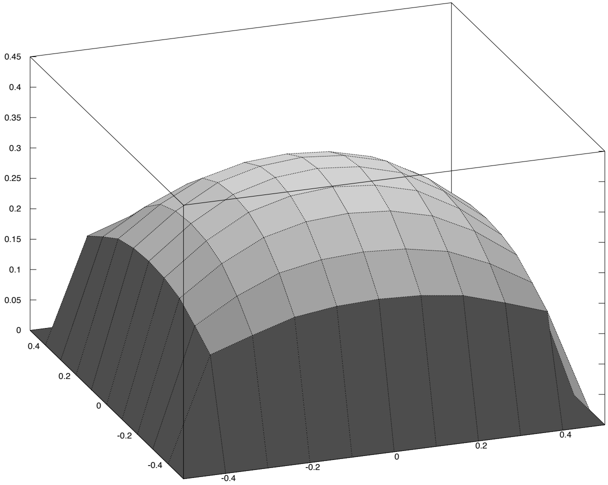
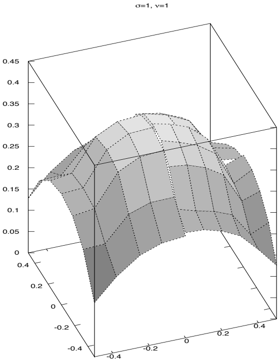
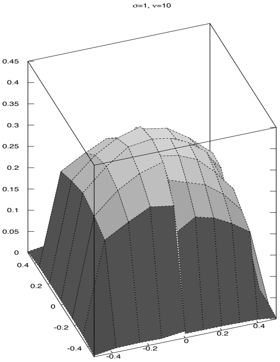
Solution u_h : conforming approximation

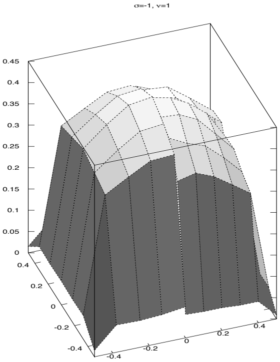


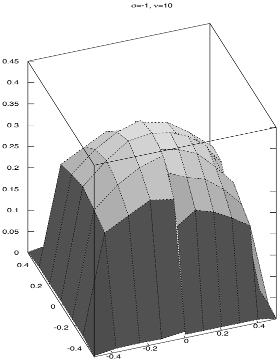
Solution u_h : symmetric Nitsche approximation



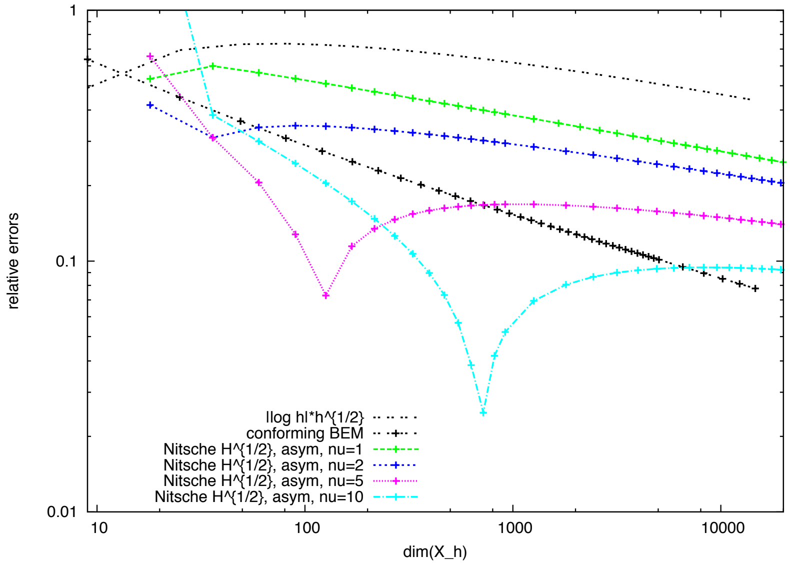


Solution u_h : skew-symmetric Nitsche approximation

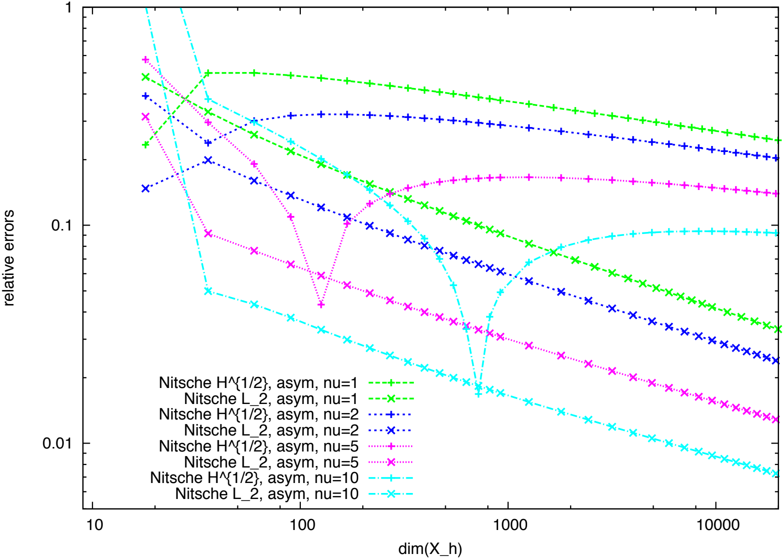




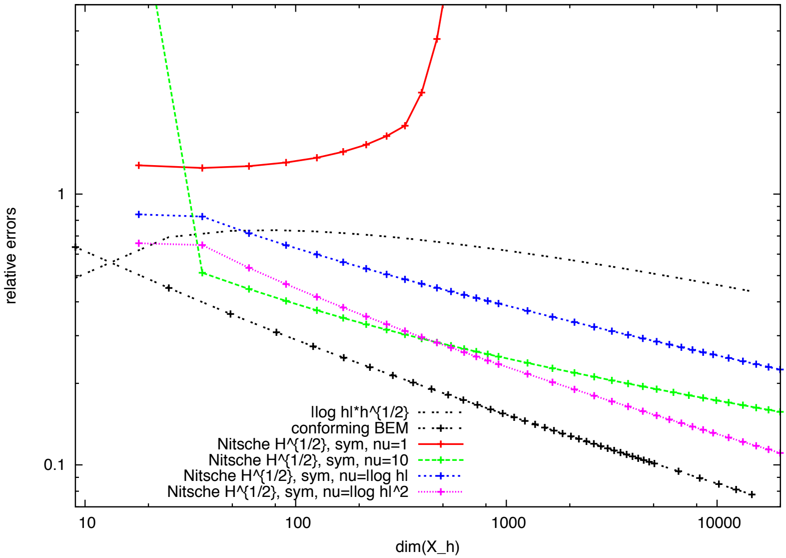
Antisymmetric Nitsche method, $H^{1/2}$-error



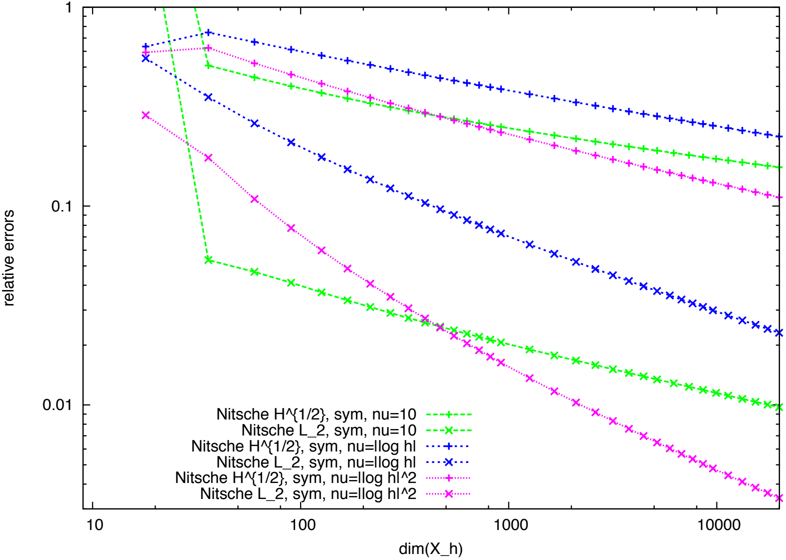
Antisymmetric Nitsche method, $H^{1/2}$ and $L_2(\gamma)$-errors



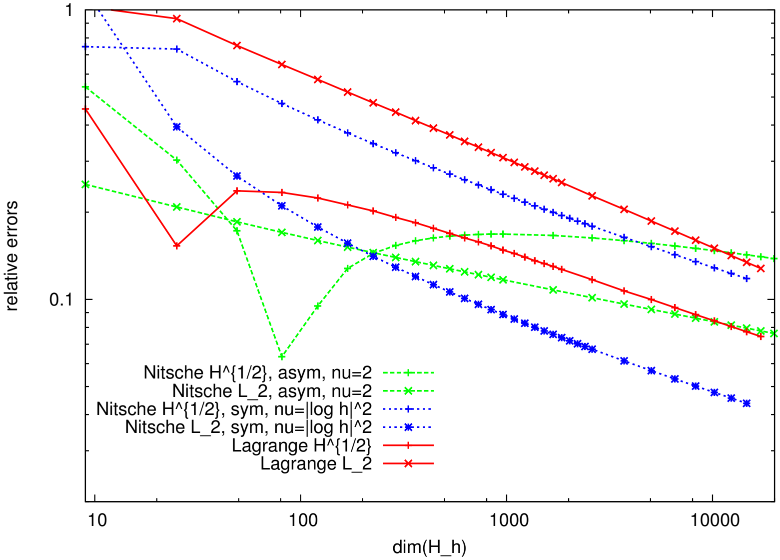
Symmetric Nitsche method, $H^{1/2}$-error



Symmetric Nitsche method, $H^{1/2}$ and $L_2(\gamma)$-errors



Comparing Nitsche and Lagrange



Conclusions

- simple coupling procedure
- almost quasi-optimal convergence
- standard linear solvers
- no additional unknowns

Open problems

- non-uniform meshes and
- higher order approximations

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