Optimal pressure estimate for the Crank-Nicolson scheme applied to the non-stationary Stokes problem

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Flow model

- **non-stationary incompressible $\{$Navier–$\}$Stokes equations:**

Find $u : \Omega \times [0,T] \to \mathbb{R}^d$ and $p : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} \partial_t u - \nu \triangle u + \Big\{ (u \cdot \nabla) u \Big\} + \nabla p & = & f & \text{in } \Omega \times (0,T)\,, \\ \operatorname{div} u & = & 0 & \text{in } \Omega \times [0,T]\,, \\ u & = & g & \text{on } \Gamma \times [0,T]\,, \\ u & = & u_0 & \text{on } \Omega \times \{0\}\,. \end{array} \right.$$

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- **u-problem:** Find $t \mapsto u(t) \in V^{\text{div}}$ s.t. $\forall\, v \in V^{\text{div}},\ t \in (0,T)$

$$\begin{aligned} (d_t u(t), v) + \nu\,(\nabla u(t), \nabla v) + ((u(t) \cdot \nabla) u(t), v) &= (f(t), v) \\ u(0) &= u_0 \end{aligned}$$

Result of Heywood & Rannacher from 1990

- **discretely div-free FE space:**

$$V_h^{\mathrm{div}} := \{v_h \in V_h : \ (\operatorname{div} v_h, q_h) = 0 \ \forall q_h \in Q_h\} \approx V^{\mathrm{div}}$$

- **Crank-Nicolson-scheme:** Find $(U_n, P_n)_{n=1}^N \subset V_h^{\mathrm{div}} \times Q_h$ such that for all $v_h \in V_h$

$$(\bar{\partial}_t U_n, v_h) + a(\bar{U}_n, v_h) + \Big\{ b(\bar{U}_n, \bar{U}_n, v_h) \Big\} - (P_n, \operatorname{div} v_h) = (\bar{f}_n, v_h)$$

where

$$\bar{\partial}_t U_n := \frac{U_n - U_{n-1}}{\tau_n}, \qquad \bar{U}_n := \frac{1}{2}(U_n + U_{n-1})$$

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- **they have proved:** e.g. for $(V_h, Q_h) = (\mathbb{Q}_m, \mathbb{P}_{m-1}^{dc})$

$$\begin{aligned} \|U_n - u(t_n)\|_{L^2(\Omega)} &\leq C(\tau^2 + h^{m+1}) \\ \|P_n - p(t_n)\|_{L^2(\Omega)} &\leq C(\tau + h^m) \end{aligned}$$

Numerical example [Hussain, Sch., Turek '12]

- prescribed velocity and pressure on $\Omega = (0,1)^2$

$$\begin{aligned} u_1(x,y,t) &:= x^2(1-x)^2\left[2y(1-y)^2 - 2y^2(1-y)\right]\sin(10\pi t), \\ u_2(x,y,t) &:= -\left[2x(1-x)^2 - 2x^2(1-x)\right]y^2(1-y)^2\sin(10\pi t), \\ p(x,y,t) &:= -(x^3+y^3-0.5)(1.5+0.5\sin(10\pi t)). \end{aligned}$$

- $(V_h, Q_h) = (\mathbb{Q}_2, \mathbb{P}_1^{dc})$, $h = 2^{-6}$, $\|\cdot\|_2$ via Gauss-1 formula ($\bar{P}_n$)

$$\|e\|_2 := \max_n \|e(t_n)\|_{L^2(\Omega)}, \qquad \tilde{p}_{h,\tau}(t_n) := \frac{1}{2}(\bar{P}_n + \bar{P}_{n+1})$$

$1/\tau$	$\|p - p_{h,\tau}\|_2$	EOC	$\|p - \tilde{p}_{h,\tau}\|_\infty$	EOC	$\|u - u_{h,\tau}\|_\infty$	EOC
20	4.25E-02		1.00E-01		8.17E-04	
40	1.08E-02	1.97	2.94E-02	1.77	2.10E-04	1.96
80	2.73E-03	1.99	7.63E-03	1.94	5.13E-05	2.03
160	6.83E-04	2.00	1.93E-03	1.99	1.28E-05	2.00
320	1.71E-04	2.00	4.83E-04	2.00	3.20E-06	2.00
640	4.39E-05	1.96	1.21E-04	2.00	8.01E-07	2.00

Crank-Nicolson as cGP(1)-Gauss(1)-method

$$u_{\tau,h} \in X^u_{\tau,h} := \{u_h \in C(I, V_h^{\mathrm{div}}) : \; u_h|_{I_n} \in \mathbb{P}_1(I_n, V_h^{\mathrm{div}})\}, \quad I = [t_0, t_N]$$

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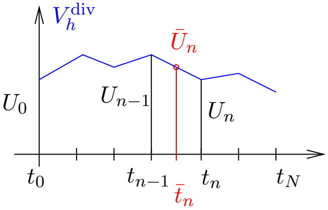

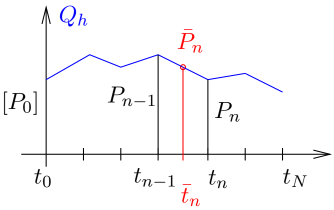

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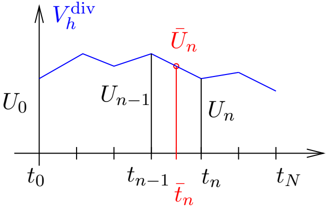

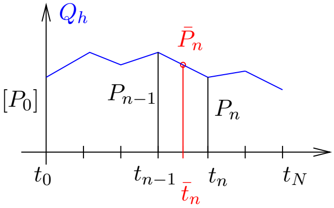

$$\int_{I_n} \Big\{ (d_t u_{\tau,h}, v_h) + a(u_{\tau,h}, v_h) + b(v_h, p_{\tau,h}) \Big\} dt = \int_{I_n} (f(t), v_h)\, dt \quad \forall v_h \in V_h$$

$$\Rightarrow \quad (d_t \bar{U}_n, v_h) + a(\bar{U}_n, v_h) + b(v_h, \bar{P}_n) = (\bar{f}_n, v_h) \qquad \forall v_h \in V_h$$

we will prove: $\boxed{\|\bar{P}_n - p(\bar{t}_n)\|_{L^2(\Omega)} \le C(\tau^2 + h^m)}$

link to the proof

- the proof is still unpublished
- a preprint will be availabe in the near future at:

`http://www-ian.math.uni-magdeburg.de/home/schieweck/papers.html`

Summary and outlook

- the **computed pressure** in the Crank-Nicolson scheme for the Stokes problem is

$$\bar{P}_n = p_{\tau,h}(\bar{t}_n), \qquad \bar{t}_n = \frac{1}{2}(t_{n-1} + t_n)$$

- it holds (if $p_{\tau,h}(t) \sim \mathbb{P}^{dc}_{m-1}$) the **optimal error estimate**

$$\|p(\bar{t}_n) - \bar{P}_n\|_{L^2(\Omega)} \leq C(\tau^2 + h^m)$$

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- **conjecture:** result can be generalized to higher order cGP(2)-method where

$$u_{\tau,h}|_{I_n} \in \mathbb{P}_2(I_n, V_h), \qquad p_{\tau,h}|_{I_n} \in \mathbb{P}_2(I_n, Q_h),$$

  numerical experiments [Hussain, Sch., Turek '12] show

$$\|p(t_n) - R^{\text{post}}(p_{\tau,h})(t_n)\|_{L^2(\Omega)} \leq C(\tau^4 + h^m)$$