Sharp controllability results for conservative PDEs

Hayk Nersisyan (BCAM)

December 10, 2012

We consider the generic linear control problem

$$\begin{aligned} \dot{y} &= Ay(t) + u(t)b && (1a) \\ y(0) &= 0 && (1b) \\ y(T) &= y_1 && (1c) \end{aligned}$$

1. A is a skew adjoint on $\mathcal{H}$ (this implies that the spectrum $\sigma(A)$ is purely imaginary: $\sigma(A) \subset i\mathbb{R}$)

[1] We use the symbol "$\asymp$" if inequalities from below and above hold, such as for equivalent norms etc

1. A is a skew adjoint on $\mathcal{H}$ (this implies that the spectrum $\sigma(A)$ is purely imaginary: $\sigma(A) \subset i\mathbb{R}$)
2. the spectrum of A is discrete, $\sigma(A) = \{i\lambda_n \mid n \in \mathbb{N}\}$ ($\lambda_n \in \mathbb{R}$), and the eigenfunctions $\{\phi\}_{n\in\mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$.

[1] We use the symbol "$\asymp$" if inequalities from below and above hold, such as for equivalent norms etc

1. A is a skew adjoint on $\mathcal{H}$ (this implies that the spectrum $\sigma(A)$ is purely imaginary: $\sigma(A) \subset i\mathbb{R}$)
2. the spectrum of A is discrete, $\sigma(A) = \{i\lambda_n \mid n \in \mathbb{N}\}$ ($\lambda_n \in \mathbb{R}$), and the eigenfunctions $\{\phi\}_{n\in\mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$.
3. all eigenvalues are simple

[1] We use the symbol "$\asymp$" if inequalities from below and above hold, such as for equivalent norms etc

1. A is a skew adjoint on $\mathcal{H}$ (this implies that the spectrum $\sigma(A)$ is purely imaginary: $\sigma(A) \subset i\mathbb{R}$)
2. the spectrum of A is discrete, $\sigma(A) = \{i\lambda_n \mid n \in \mathbb{N}\}$ ($\lambda_n \in \mathbb{R}$), and the eigenfunctions $\{\phi\}_{n\in\mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$.
3. all eigenvalues are simple
4. there exist $p > 0$, $C_1, C_2 > 0$ such that

$$C_1 n^p \leq \lambda_n \leq C_2 n^p \qquad (\text{as } n \to \infty)$$

We abbreviate this as $\lambda_n \asymp n^p$ $(n \to \infty)$[1].

[1]We use the symbol "$\asymp$" if inequalities from below and above hold, such as for equivalent norms etc

1. A is a skew adjoint on $\mathcal{H}$ (this implies that the spectrum $\sigma(A)$ is purely imaginary: $\sigma(A) \subset i\mathbb{R}$)
2. the spectrum of A is discrete, $\sigma(A) = \{i\lambda_n \mid n \in \mathbb{N}\}$ ($\lambda_n \in \mathbb{R}$), and the eigenfunctions $\{\phi\}_{n\in\mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$.
3. all eigenvalues are simple
4. there exist $p > 0,\ C_1, C_2 > 0$ such that

$$C_1 n^p \leq \lambda_n \leq C_2 n^p \qquad (\text{as } n \to \infty)$$

   We abbreviate this as $\lambda_n \asymp n^p$ $(n \to \infty)$[1].
5. $b \in \mathcal{H}$
6. $\gamma_n := \langle b, \phi_n \rangle \neq 0$ $(\forall n \in \mathbb{N}_0)$.
7. there exist $q \in \mathbb{R}$ such that

$$\gamma_n \asymp n^{-q} \qquad (n \to \infty).$$

[1] We use the symbol "$\asymp$" if inequalities from below and above hold, such as for equivalent norms etc

Let us choose the state and control spaces as

$$\begin{aligned}\mathcal{H}^\alpha &= H^\alpha(\{\phi_n\}) = \left\{\sum_n c_n\phi_n \mid (c_n) \in h^\alpha\right\} \quad \text{and} \\ \mathcal{U}^\beta &= H_0^\beta(0,T;\mathbb{C}) \\ &= \left\{u \in H^\beta(0,T;\mathbb{C}) \mid u^{(k)}(0) = u^{(k)}(T) = 0 (k = 0, 1, \dots, \beta - 1)\right\}\end{aligned}$$

Here

$$h^s := \{(c_n)_{n\in\mathbb{N}} \subset \mathbb{C} \mid (\lambda_n^s c_n) \in \ell^2\}$$

as usual.

Case $p > 1$

Case $p > 1$

In this case controllability holds in any time $T > 0$. The following relationship between the indices α and β will mark the dividing line between controllability and non-controllability.

$$\alpha = p\beta + q. \tag{2}$$

Theorem

Let $p > 1$, q, α, β satisfying (2), and let $T > 0$. Then, for every $y_1 \in \mathcal{H}^\alpha$, there exists a control function $u \in \mathcal{U}^\beta$ such that $\Phi_T(u) = y_1$.

By standard calculations we see that the solution to (1a)-(1b) is given by

$$y(t) = \sum_n \left(\int_0^t e^{i\lambda_n(t-s)} u(s) ds \right) < b, \phi_n > \phi_n. \qquad (3)$$

By standard calculations we see that the solution to (1a)-(1b) is given by

$$y(t) = \sum_n \left(\int_0^t e^{i\lambda_n (t-s)} u(s) ds \right) < b, \phi_n > \phi_n. \qquad (3)$$

1. The exponential system $\{e^{-i\lambda_n t}\}_{n\in\mathbb{N}}$ forms a Bessel sequence in $L^2(0, T)$.
2. The functions $\{e^{-i\lambda_n t}\}_{n\in\mathbb{N}}$ form a Riesz-Fischer sequence in $L^2(0, T)$.

Then there exists a solution $u \in L^2(0, T; \mathbb{C})$ of the moment problem

$$\int_0^T e^{-i\lambda_n t} u(t) dt = e_n, \qquad n \in \mathbb{N} \tag{4}$$

that satisfies $u \in \mathcal{U}^\beta = H_0^\beta(0, T; \mathbb{C})$ and

$$\|u\|_{\mathcal{U}^\beta} \lesssim \|(\lambda_n^\beta e_n)_{n \in \mathbb{N}}\|_{\ell^2}. \tag{5}$$

As usual, we turn the controllability problem into a moment problem. Writing

$$y_1 = \sum_n d_n \phi_n,$$

the condition $\Phi_T(u) = y(T) = y_1$ takes the form

$$\int_0^T e^{-i\lambda_n s} u(s) ds = \frac{d_n e^{-i\lambda_n T}}{\gamma_n} =: e_n, \qquad n \in \mathbb{N} \tag{6}$$

(where 6 allows us to divide by γ_n). Now:

$$y_1 \in \mathcal{H}^\alpha \quad \Rightarrow \quad (d_n) \in h^\alpha \quad \stackrel{7}{\Rightarrow} \quad (e_n) \in h^{\alpha-q} \quad \Rightarrow \quad (\lambda_n^{\alpha-q} e_n) \in \ell^2$$

Case $p \leq 1$

Theorem

Let $p \leq 1$, $\alpha > 0$ and let $T = \infty$. Then, for every $y_1 \in \mathcal{H}^\alpha$, there exists a control function $u \in \mathcal{U}^\beta$ such that $\lim_{t\to\infty} y(t) = y_1$.

Non-controllability.

The main tool for proving non-controllability is the Kolmogorov ε-entropy

Definition

Let $\mathcal{X}$ be a Banach space. For any compact subset $K \subset \mathcal{X}$ and $\varepsilon > 0$, we denote by $N_\varepsilon(K, \mathcal{X})$ the minimal number of sets of diameters $\leq 2\varepsilon$ required to cover K. The *Kolmogorov ε-entropy* $H_\varepsilon(K, \mathcal{X})$ of K is defined as $H_\varepsilon(K, \mathcal{X}) = \ln N_\varepsilon(K, \mathcal{X})$

We'll need the following properties.

Proposition

(i) Let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$, $r \in \mathbb{R}$, and $\delta > 0$. Then for any ball $B \subset H^{r+\delta}(\Omega)$

$$H_\varepsilon(B, H^r(\Omega)) \asymp \left(\frac{1}{\varepsilon}\right)^{d/\delta}$$

(ii) Let $\mathcal{Y}$ be another Banach space and let $f : K \to \mathcal{Y}$ be a Hölder continuous function:

$$\|f(u_1) - f(u_2)\|_{\mathcal{Y}} \leq L\|u_1 - u_2\|_{\mathcal{X}}^\theta$$

for all $u_1, u_2 \in K$ and for some constants $L > 0$ and $\theta \in (0, 1)$. Then

$$H_\varepsilon(f(K), \mathcal{Y}) \leq H_{(\varepsilon/L)^{1/\theta}}(K, \mathcal{X}).$$

To formulate the main result of this section we define the sets of states that are attainable in finite and infinite time, respectively:

$$\mathcal{A}_T^{\alpha,\beta} := \{H^\alpha(\Omega) \ni \Phi_T(u) \mid u \in H_0^\beta(0,T)\} \quad \text{and} \quad \mathcal{A}^{\alpha,\beta} := \bigcup_{T>0} \mathcal{A}_T^{\alpha,\beta}.$$

Theorem

Let $p, q, \alpha, \beta \in \mathbb{R}$ satisfy (2). Then the set of reachable states $\mathcal{A}^{\alpha,\beta+}$ does not contain any open ball of $H^\alpha(\Omega)$.

We assume that there is a ball $B \subset H^\alpha$ such that $\Phi_T(Q) = B$, where $Q \subset H^{\beta+\delta_\beta}$. Using Proposition 4 (for $K = Q, \mathcal{X} = H^\beta, \mathcal{Y} = H^\alpha$), we obtain for any $\gamma > 0$

$$\left(\frac{1}{\varepsilon}\right)^{d/\gamma} \asymp H_\varepsilon(\underbrace{\Phi_T(Q)}_{\subset H^\alpha(\Omega)}, H^{\alpha-\gamma}(\Omega)) \precsim H_{\varepsilon/L}(\underbrace{Q}_{\subset H_0^{\beta+\delta_\beta}(0,T)}, H_0^\beta(0,T)) \asymp \left(\frac{L}{\varepsilon}\right)^{1/\delta_\beta}.$$

This gives a contradiction for $\varepsilon \to 0^+$ if

$$\frac{d}{\gamma} > \frac{1}{\delta_\beta} \iff d\delta_\beta > \gamma.$$

Thank you for your attantion