# Pullback Exponential Attractors for Evolution Processes in Banach Spaces [1]

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[1] joint work with Alexandre N. Carvalho

# Plan

# Introduction

Autonomous initial value problems

$$\begin{cases} \frac{\partial}{\partial t}u = A(u) & \text{in Banach space } X \\ u|_{t=0} = u_0 & u_0 \in X \end{cases} \tag{1}$$

Assuming global well-posedness of (1)

$\Longrightarrow$ generates a semigroup in $X$, $S(t) : X \to X$, $t \geq 0$,

$$\begin{aligned} S(t) \circ S(s) &= S(t+s) \qquad t, s \geq 0 \\ S(0) &= Id \\ (t, x) &\mapsto S(t)x \qquad \text{continuous} \end{aligned}$$

# Longtime Dynamics

Many dissipative systems posses a global attractor $\mathcal{A}$

(a) $\emptyset \neq \mathcal{A} \subset X$ compact,

(b) $S(t)\mathcal{A} = \mathcal{A}$, $t \geq 0$, invariant,

(c) attracts every bounded set $D \subset X$,

$$\lim_{t\to\infty} \mathrm{dist}_H(S(t)D, \mathcal{A}) = 0.$$

*Properties:*

$\mathcal{A}$ unique, minimal closed set attracting all bounded sets,
in most cases, the fractal dimension $\dim_f(\mathcal{A}) = k < \infty$
$\Longrightarrow$ there exists a continuous embedding $\mathcal{A} \hookrightarrow \mathbb{R}^{2\lfloor k \rfloor + 1}$

# Drawbacks of Global Attractors

Example:

$$\begin{aligned} \dot{x} &= -x(x-1)^2 \\ x|_{t=0} &= x_o,\ x_o \in \mathbb{R} \qquad \Longrightarrow \qquad \mathcal{A} = [0,1], \end{aligned}$$

but $\forall \epsilon > 0$

$$\begin{aligned} \dot{x} &= -(x(x-1)^2 + \epsilon) \\ x|_{t=0} &= x_o,\ x_o \in \mathbb{R} \qquad \Longrightarrow \qquad \mathcal{A}_\epsilon = \{y_\epsilon\}. \end{aligned}$$

In general,

- rate of convergence unknown, can be arbitrarily slow
- $\mathcal{A}$ not stable under perturbations
- complex structure, not computable

# Exponential Attractors [1, 2]

$\mathcal{M}$ is an exponential attractor for $S(t), t \geq 0$, if

(a) $\emptyset \neq \mathcal{M} \subset X$ compact,

(b) $\dim_f(\mathcal{M}) < \infty$ finite dimensional,

(c) $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$, semi-invariant,

(d) $\exists \omega > 0$ such that for every bounded set $D \subset X$

$$\lim_{t\to\infty} e^{\omega t} \mathrm{dist}_H(S(t)D, \mathcal{M}) = 0.$$

Recently, extension to non-autonomous problems ...

[1] **Eden, Foias, Nicolaenko, Temam (1994).**

[2] **Efendiev, Miranville, Zelik (2000).**

# Evolution Processes

Nonautonomous initial value problems

$$\begin{cases} \frac{\partial}{\partial t} u = A(t)u & \text{in Banach space } X \\ u|_{t=s} = u_s & s \in \mathbb{R}, u_s \in X \end{cases} \tag{2}$$

Assuming global well-posedness of (2)

$\Longrightarrow$ generates an evolution process in $X$, $U(t,s) : X \to X$, $t \geq s$, family of operators such that

$$\begin{aligned} U(t,s) \circ U(s,r) &= U(t,r) && t \geq s \geq r \\ U(t,t) &= Id && t \in \mathbb{R} \\ (t,s,x) &\mapsto U(t,s)x && \text{continuous} \end{aligned}$$

# Nonautonomous Attractors

*Different concepts:*
uniform attractors, pullback and forwards attractors

$\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ global pullback attractor for $U(t,s), t \geq s$, if

(a) $\emptyset \neq \mathcal{A}(t) \subset X$ compact for all $t \in \mathbb{R}$,

(b) $U(t,s)\mathcal{A}(s) = \mathcal{A}(t)$, $t \geq s$ invariant,

(c) for all bounded $D \subset X$ and $t \in \mathbb{R}$

$$\lim_{s\to\infty} \mathrm{dist}_H(U(t,t-s)D, \mathcal{A}(t)) = 0,$$

(d) $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ minimal within the family of closed subsets, that pullback attract all bounded sets.

# Exponential Pullback Attractors

Global pullback attractors have similar drawbacks as global attractors in the autonomous case.

$\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ exponential pullback attractor for $U(t,s), t \geq s$, if

(a) $\emptyset \neq \mathcal{M}(t) \subset X$ compact for all $t \in \mathbb{R}$,

(b) $\sup_{t\in\mathbb{R}} \dim_f(\mathcal{M}(t)) < \infty$ finite dimensional,

(c) $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$, $t \geq s$ semi-invariant,

(d) $\exists\omega > 0$ such that for every bounded $D \subset X$ and $t \in \mathbb{R}$

$$\lim_{s\to\infty} e^{\omega s} \text{dist}_H(U(t,t-s)D, \mathcal{M}(t)) = 0.$$

# Exponential Pullback Attractors [3,4,5,6]

- exponential rate of convergence, robust under perturbations
- finite dimensional, $\mathcal{A}(t) \subset \mathcal{M}(t) \Longrightarrow$ existence and finite fractal dimension of $\{\mathcal{A}(t), t \in \mathbb{R}\}$
- However, not unique !

Assumptions in [4,5]:

- process satisfies smoothing property
- Hölder continuous in time
- strongly pullback bounded dissipative $\Longrightarrow$

$$\bigcup_{s \leq t} \mathcal{A}(s) \subset \bigcup_{s \leq t} \mathcal{M}(s) \quad \text{is bounded } \forall t \in \mathbb{R}$$

[3] **Efendiev, Miranville, Zelik (2005).**
[4] **Langa, Miranville, Real (2010).**
[5] **Czaja, Efendiev (2011).**
[6] **Efendiev, Yamamoto, Yagi (2011).**

# Assumptions

$U(t,s), t \geq s$, evolution process in Banach space $X$, $U = S + C$

(A$_1$) $Y$ normed space, $X \hookrightarrow\hookrightarrow Y$ compact, dense

(A$_2$) $\{B(t) | t \in \mathbb{R}\} \subset X$ bounded, semiinvariant, *pullback absorbing sets*: For every bounded $D \subset X, t \in \mathbb{R}$ exists $T_{D,t} > 0$:
$$U(r, r-s)D \subset B(r) \qquad \forall\, s \geq T_{D,t}, r \leq t$$

(A$_3$) (*S smoothing*) There exists $\tilde{t} > 0$ and $\kappa > 0$:
$$\|S(t+\tilde{t}, t)u - S(t+\tilde{t}, t)v\|_X \leq \kappa \|u - v\|_Y \qquad \forall u, v \in B(t)$$

(A$_4$) (*C contraction*) There exists $0 \leq \lambda < \frac{1}{2}$:
$$\|C(t+\tilde{t}, t)u - C(t+\tilde{t}, t)v\|_X \leq \lambda \|u - v\|_X \qquad \forall u, v \in B(t)$$

(A$_5$) (*U Lipschitz continuous*) For all $s \geq t$ there exists $L_{t,s} > 0$:
$$\|U(s,t)u - U(s,t)v\|_X \leq L_{t,s} \|u - v\|_X \qquad \forall u, v \in B(t)$$

# Existence of Pullback Exponential Attractors

## Theorem

*Let $U(t,s), t \geq s$, be an evolution process in $X$ and $(A_1)$-$(A_5)$ be satisfied. If the diameter of the absorbing sets $\{B(t), t \in \mathbb{R}\}$ grows at most sub-exponentially in the past, then, there exists for every $0 < \nu < \frac{1}{2} - \lambda$ a pullback exponential attractor $\{\mathcal{M}^\nu(t) | \ t \in \mathbb{R}\}$ and*

$$\dim_f^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left( N^Y_{\frac{\nu}{\kappa}}(B^X_1(0)) \right) \qquad \forall t \in \mathbb{R},$$

*where $N^Z_\epsilon(D)$ is the minimal number of $\epsilon$-balls in a metric space $Z$ with centers in $D$, needed to cover $D \subset Z$.*

# Properties

- It follows the existence of the global pullback attractor and

$$\dim_f^V(\mathcal{A}(t)) \leq \inf_{\nu\in(0,\frac{1}{2}-\lambda)} \left\{ \log_{\frac{1}{2(\nu+\lambda)}} \left( N^Y_{\frac{\nu}{\kappa}}(B^X_1(0)) \right) \right\}.$$

  Remark: Hypothesis $(\mathcal{H}_4)$ can be omitted.
- Relation with the global pullback attractor

$$\mathcal{M}(t) = \Lambda(\hat{B}, t) \cup \bigcup_{n\in\mathbb{N}} E_n(t), \qquad \sharp\{E_n(t)\} < \infty.$$

- If the pullback absorbing time $T_{D,t}$ is independent of $t$, the pullback exponential attractor $\mathcal{M}(t)$ is also a forwards exponential attractor.

# Applications

## Nonautonomous Chafee Infante Equation

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + \lambda u(x,t) - \beta(t)\big(u(x,t)\big)^3 && x \in \Omega,\ t > s \\ \frac{\partial}{\partial \nu} u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s \\ u(x,s) &= u_s(x) && x \in \Omega,\ s \in \mathbb{R} \end{aligned}$$

$\Omega \subset \mathbb{R}^n$ bounded, smooth domain, $\lambda \in \mathbb{R}$, $u_s \in C_0(\overline{\Omega})$.

$\beta \in C^1(\mathbb{R};\mathbb{R}_+)$,

$$0 < \beta(t) \leq \beta_0, \qquad \frac{|\beta'(t)|}{\beta(t)} \leq \beta_1 \qquad \forall t \in \mathbb{R},$$

$$\lim_{t\to-\infty} \beta(t) = 0, \qquad \lim_{t\to-\infty} \frac{e^{\gamma t}}{\beta(t)} = 0 \qquad \forall \gamma > 0.$$

# Applications

## Nonautonomous Damped Wave Equation

$$\begin{aligned} u_{tt}(x,t) + \beta(t) u_t(x,t) &= \Delta u(x,t) + f(u(x,t)) && x \in \Omega,\ t > s \\ u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s \\ u(x,s) = u_s(x), \quad u_t(x,s) &= v_s(x) && x \in \Omega,\ s \in \mathbb{R} \end{aligned}$$

$\Omega \subset \mathbb{R}^n$ bounded, smooth domain, $u_s \in H_0^1(\Omega), v_s \in L^2(\Omega)$.

$\beta \in C^1(\mathbb{R}; \mathbb{R}_+)$, $f \in C^1(\mathbb{R}; \mathbb{R})$,

$$\limsup_{z \to \infty} \frac{f(z)}{z} = 0, \quad |f'(z)| \leq c(1 + |z|^\rho), \quad 0 < \rho < \frac{2}{n-2},$$

$$0 < \beta_0 \leq \beta(t) \leq \beta_1, \quad \forall t \in \mathbb{R}.$$