Wave nonlinearity expansion of $\lambda\phi^4$ theory

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We consider the nonlinear wave equation with a real scalar field,

$$\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi + \frac{\lambda}{6}\Phi^3 = 0,$$

where λ is a positive coupling constant. Based on the wave turbulence formalism, we will present an approach to the study of this model, starting from its homogeneous (not dependent on the spatial coordinates) periodic solutions:

$$\phi_T(t) = \sqrt{\frac{6}{\lambda}}\frac{4K(1/2)}{T}\text{cn}\left(\frac{4K(1/2)t}{T}, \frac{1}{2}\right), \quad T \text{ is the period.} \qquad (1)$$

Here $K(1/2)$ is the complete elliptic integral of the first kind. The corresponding energy density ρ is nonzero,

$$\rho = \frac{1}{2}\dot{\Phi}_T^2 + \frac{\lambda}{24}\Phi_T^4 = \frac{384K(1/2)^4}{\lambda T^4}.$$

These solutions include non-perturbative effects due to anharmonicities.

The Hamiltonian structure of the model is

$$\begin{aligned} H &= \int dt\, d^d x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{\lambda}{24}\Phi^4 \right), \\ \partial_t \phi &= \frac{\delta H}{\delta \Pi} = \Pi, \qquad \partial_t \Pi = -\frac{\delta H}{\delta \Phi} = \nabla^2\Phi - \frac{\lambda}{6}\Phi^3. \end{aligned} \tag{2}$$

- In order to include the spatial dependence one performs the plane wave decomposition

$$\Phi(t, \boldsymbol{x}) = \phi_T(t) + \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

  so the main task will be the study of the equations satisfied by the Fourier modes $\varphi_{\boldsymbol{k}}(t)$.
- Therefore, the first step is to discuss the role of the quadratic time-dependent Hamiltonian

$$H_2(t) = \int dt \sum_{\boldsymbol{k}} \frac{1}{2} \left[\Pi^*_{\boldsymbol{k}} \Pi_{\boldsymbol{k}} + \left(k^2 + \frac{\lambda \Phi_T(t)^2}{2} \right) \varphi^*_{\boldsymbol{k}} \varphi_{\boldsymbol{k}} \right]. \tag{3}$$

We first consider a single mode $\boldsymbol{k}$. (Note that $\varphi^*_{\boldsymbol{k}} = \varphi_{-\boldsymbol{k}}$).

At quadratic level one mode is described by the time-dependent periodic Hamiltonian

$$\mathcal{H}_2 = \frac{1}{2}\phi^T H(t)\phi, \qquad H(t) = \begin{pmatrix} K(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad K(t+T) = K(t), \tag{4}$$

where the (real) quantities $\phi^T = (\varphi, \pi)$ are canonically conjugated, and $K(t)$ is a periodic function of period T. The equations of motion arises from

$$J\dot{\phi} = H\phi, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J^{-1} = J^T = -J, \tag{5}$$

and they leads to Hill's equation for $\varphi(t)$:

$$\ddot{\varphi} + K(t)\varphi = 0. \tag{6}$$

In matrix notation we have the system

$$\phi = \begin{pmatrix} \varphi \\ \pi \end{pmatrix}, \qquad \dot{\phi} = \Omega(t)\phi = \begin{pmatrix} 0 & 1 \\ -K(t) & 0 \end{pmatrix}\phi, \qquad K(t) = k^2 + \frac{\lambda}{2}\phi_T^2. \tag{7}$$

The fundamental matrix solution of (7) we will take verifies

$$\Phi(0) = 1_{2\times 2}.$$

Since $\Phi(t+T) = \Phi(t)\Xi$ for some non-singular matrix Ξ (the monodromy matrix), one has $\Xi = \Phi(T)$. This matrix can be diagonalized, and we will assume that for some values of k, its eigenvalues come in a conjugate pair (of Floquet indices),

$$P^{-1}\Xi P = \Lambda = \mathrm{diag}\{e^{i\omega T}, e^{-i\omega T}\}, \qquad \omega > 0. \tag{8}$$

The stability angle ωT is a function of the wavenumber k. It is convenient to define the diagonal matrix $B = \mathrm{diag}\{i\omega, -i\omega\}$, in such a form that $\Lambda = \exp(BT)$. By introducing a new matrix $\Upsilon(t)$ through

$$\Phi(t) = \Upsilon(t)P^{-1}, \tag{9}$$

we see that the property $\Phi(t+T) = \Phi(t)\Xi$ adopts the form $\Upsilon(t+T) = \Upsilon(t)e^{BT}$. Therefore we write $\Upsilon(t) = \Gamma(t)e^{Bt}$, where now $\Gamma(t)$ is a periodic matrix, $\Gamma(t+T) = \Gamma(t)$. It obeys the equation

$$\dot{\Gamma} + \Gamma B = \Omega\Gamma. \tag{10}$$

The Floquet-Lyapunov transformation consists in the time-dependent coordinate change $\Phi \to \chi$ given by

$$\phi(t) = \Gamma(t)\chi(t), \tag{11}$$

which, in virtue of Eq. (10), gives rise to

$$\dot{\chi} = B\chi, \qquad \chi = \begin{pmatrix} \mathcal{A}(t)^* \\ \mathcal{A}(t) \end{pmatrix}. \tag{12}$$

Since B is diagonal with constant coefficients, the solutions are

$$\mathcal{A}(t) = \mathcal{A}(0)e^{-i\omega t}, \qquad \mathcal{A}(t)^* = \mathcal{A}(0)^* e^{i\omega t}, \qquad . \tag{13}$$

This behavior suggests that the Hamiltonian of the system in terms of the Floquet modes must be written as

$$\mathcal{H} = \omega \mathcal{A}(t)^* \mathcal{A}(t), \tag{14}$$

where $\mathcal{A}$ is the new (complex) coordinate and $i\mathcal{A}^*$ its canonicaly conjugate momentum. Therefore the canonical form of the Hamilton's equations is

$$\begin{aligned} \frac{d\mathcal{A}}{dt} &= \frac{\partial \mathcal{H}}{\partial (i\mathcal{A}^*)} = -i\omega \mathcal{A}, \\ \frac{d(i\mathcal{A}^*)}{dt} &= -\frac{\partial \mathcal{H}}{\partial \mathcal{A}} = -\omega \mathcal{A}^*, \end{aligned} \tag{15}$$

which precisely coincides with Eq. (12).

What is the generating function of the canonical transformation $(\varphi, \pi) \to (\mathcal{A}, i\mathcal{A}^*)$?

The form of the equations arising from (10) leads to the fact that the Γ-matrix may be written in terms of two complex functions U, V:

$$\Gamma = \begin{pmatrix} U & U^* \\ V & V^* \end{pmatrix}, \qquad \Gamma^{-1} = \frac{1}{UV^* - U^*V} \begin{pmatrix} V^* & -U^* \\ -V & U \end{pmatrix}. \tag{16}$$

As a consequence of $\Gamma(t) = \Phi(t) P e^{-Bt}$, it follows that $\det \Gamma(t) = UV^* - U^*V = \det P$, since $\det \Phi(t) = 1$. The matrix P is not unique, and its determinant is not restricted in principle. However, the detailed analysis shows that the only value making the Floquet-Lyapunov transformation a canonical transformation is precisely $\det P = -i$. For this value one has $UV^* - U^*V = -i$. Therefore, it is possible to find a generating function $\mathcal{F}_1(\varphi, \mathcal{A}, t)$ obeying the consistency equations for a canonical transformation (see e. g. Goldstein *Classical Mechanics*)

$$\pi = \frac{\partial \mathcal{F}_1}{\partial \varphi}, \qquad i\mathcal{A}^* = -\frac{\partial \mathcal{F}_1}{\partial \mathcal{A}}. \tag{17}$$

The generating function adopts the simple form

$$\mathcal{F}_1(\varphi, \mathcal{A}, t) = \frac{iU^*}{2U}\mathcal{A}^2 - \frac{i}{U}\xi\mathcal{A} + \frac{V}{2U}\xi^2, \tag{18}$$

so that the new Hamiltonian reads

$$\mathcal{H} = \mathcal{H}_2 + \frac{\partial \mathcal{F}_1}{\partial t} = \omega \mathcal{A}^* \mathcal{A}. \tag{19}$$

Summarizing, the transformation

$$\begin{aligned} \varphi &= U\mathcal{A}^* + U^*\mathcal{A}, \\ \pi &= V\mathcal{A}^* + V^*\mathcal{A}, \end{aligned} \tag{20}$$

is canonical, and converts the quadratic time-dependent Hamiltonian $\mathcal{H}_2$ into $\mathcal{H} = \omega \mathcal{A}^* \mathcal{A}$. The Hamiltonian has been diagonalized!!

The fundamental matrix $\Phi(t)$, $\Phi(0) = 1_{2\times 2}$, may be expressed in terms of two independent solutions ξ, ξ^* of Hill's equation (6) as

$$\Phi(t) = \begin{pmatrix} \frac{i}{w}[\xi^*(t)\xi'(0) - \xi(t)\xi'^*(0)] & \frac{i}{w}[\xi(t)\xi^*(0) - \xi^*(t)\xi(0)] \\ -\frac{i}{w}[\xi'(t)\xi'^*(0) - \xi'^*(t)\xi'(0)] & \frac{i}{w}[\xi'(t)\xi^*(0) - \xi'^*(t)\xi(0)] \end{pmatrix}, \tag{21}$$

where the constant w corresponds to the Wronskian, $W[\xi, \xi^*] = iw$. The product $X(t)$ of two solutions satisfies the third order equation

$$X'''(t) + 4K(t)X'(t) + 2K'(t)X(t) = 0. \tag{22}$$

We may write the solutions of Hill's equation in terms of $X(t)$ and w as follows

$$\begin{aligned} \xi(t) &= \sqrt{X(t)}\exp\left(-i\int_0^t dt'\frac{w}{2X(t')}\right), \\ \xi^*(t) &= \sqrt{X(t)}\exp\left(i\int_0^t dt'\frac{w}{2X(t')}\right). \end{aligned} \tag{23}$$

Now it is easy to give formulae for the functions U and V of the canonical transformation:

$$\begin{aligned} U(t) &= -i\frac{\sqrt{X(t)}}{\sqrt{w}} \exp\left(i\int_0^t dt' \frac{w}{2X(t')}\right) e^{-i\omega t}, \\ V(t) &= \frac{w - iX'(t)}{2\sqrt{w}\sqrt{X(t)}} \exp\left(i\int_0^t dt' \frac{w}{2X(t')}\right) e^{-i\omega t}. \end{aligned} \tag{24}$$

In order to check this feature, it suffices to plug these formulae into (10). In fact, we see that when w^2 corresponds to a first integral of (22)

$$w^2 = -X'(t)^2 + 2X(t)\left[2K(t)X(t) + X''(t)\right], \tag{25}$$

the functions U and V solve the system (10).

$$\phi_T(t) = \sqrt{\frac{6}{\lambda}}\frac{4K(1/2)}{T}\mathrm{cn}\left(\frac{4K(1/2)t}{T}, \frac{1}{2}\right). \tag{26}$$

Now Hill's equation is the Lamé equation for $\ell = 2$ (see Whittaker and Watson, *Modern Analysis*):

$$\left[-\frac{d^2}{du^2} - E + \ell(\ell+1)m_0\,\mathrm{sn}^2(u, m_0)\right]\Psi = 0, \quad E = \frac{k^2T^2}{16K^2} + 1 + 4m_0.$$

A polynomial solution (Hermite-Halphen polynomial) for X:

$$X_k(t) = -576K^4\mathrm{cn}^4\left(\frac{4Kt}{T}, 1/2\right) + 24K^2T^2k^2\mathrm{cn}^2\left(\frac{4Kt}{T}, 1/2\right) + \left(576K^4 - k^4T^4\right), \tag{27}$$

$$w_k = 2k\sqrt{k^8T^8 - 1344K^4k^4T^4 + 442\,368K^8}$$

The Wronskian vanishes for

$$k = 0, \qquad kT = 2\sqrt{6}K \approx 9.083, \qquad kT = 4\,3^{1/4}K \approx 9.7604. \tag{28}$$

The angular frequency ω_k (The dispersion relation)

- A useful integral to compute ω_k is

$$\int \frac{dt}{\mathrm{sn}^2(bt,m)+a} = \frac{1}{ab}\Pi(-1/a, \mathrm{am}(bt,m), m), \tag{29}$$

where $\mathrm{am}(u,m)$ is the (Jacobian) amplitude (with the property $\mathrm{am}(2K(m),m)=\pi$), and Π is the elliptic integral of the third kind.

- The formula for the derivative with respect to the period is relatively simple

$$\frac{d(\omega_k T/2)}{dT} = \frac{k\left(12\pi k^2T^2 - k^4T^4 + \frac{98304\,\Gamma(5/4)^8}{\pi^2}\right)}{2\sqrt{k^8T^8 - 1344K^4k^4T^4 + 442\,368K^8}}. \tag{30}$$

- The asymptotics of ω_k in the infrared ($kT \ll 1$) is

$$\omega_k\frac{T}{2} = \pi + \frac{kT}{2\sqrt{3}} + O((kT)^3), \qquad kT \to 0. \tag{31}$$

$U_k(t)$ in the infrared

$$U_k(t) \sim -i\frac{3^{1/4}\mathrm{sn}(4tK/T,1/2)\,\mathrm{dn}(4tK/T,1/2)}{\sqrt{2k}}e^{-i\,2\pi t/T}, \quad kT \to 0, \tag{32}$$

Note that the period is $T/2$, as required. A more useful expression is

$$U_k(t) = \frac{T^2\sqrt{\lambda}}{2^5\,3^{1/4}\,K(1/2)^2}\left(\frac{i}{\sqrt{k}}\dot{\phi}_T(t) - \frac{\sqrt{k}}{\sqrt{3}}\phi_T(t) + \dots\right)e^{-i\,2\pi t/T} \tag{33}$$

This will be crucial.

Other useful results

- Some identities.

$$\begin{aligned} \mathrm{cn}^4(u,1/2) &= -\frac{1}{3}\frac{d^2}{du^2}\mathrm{cn}^2(u,1/2)+\frac{1}{3}, \\ \mathrm{cn}^6(u,1/2) &= \frac{1}{30}\frac{d^4}{du^4}\mathrm{cn}^2(u,1/2)+\frac{3}{5}\mathrm{cn}^2(u,1/2). \end{aligned} \tag{34}$$

- Consequences.

$$\begin{aligned} \phi_T(t)\dot{\phi}_T^3(t) &= -\frac{1}{60\lambda}\frac{d^5}{dt^5}\phi_T^2(t)+\frac{1536K(1/2)^4}{5T^4\lambda}\frac{d}{dt}\phi_T^2(t), \\ \phi_T(t)^4 &= -\frac{2}{\lambda}\frac{d^2}{dt^2}\phi_T^2(t)+\frac{3072K(1/2)^4}{\lambda^2T^4}. \end{aligned} \tag{35}$$

Fourier series

$$\begin{aligned} \mathrm{cn}^2(u,1/2) = & \frac{E(1/2)-\frac{1}{2}K(1/2)}{\frac{1}{2}K(1/2)} \\ & + \frac{2\pi^2}{\frac{1}{2}K(1/2)^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\left(\frac{n\pi}{K(1/2)}u\right), \\ q = & e^{-\pi} \approx 0.0432, \end{aligned} \tag{36}$$

$E(m)$ is the complete elliptic integral of the second kind:

$$\begin{aligned} E(m) &= \int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}d\theta, \\ K(m) &= \int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}d\theta. \end{aligned}$$

Now we return to the nonlinear wave equation.

- Plug the system into a box with periodic conditions.

$$\frac{1}{V}\sum_{\mathbf{k}} \Leftrightarrow \int \frac{d\mathbf{k}}{(2\pi)^3}, \qquad \frac{V}{(2\pi)^3}\delta_{\mathbf{k}\mathbf{k}'} \Leftrightarrow \delta(\mathbf{k}-\mathbf{k}'), \quad V \to \infty.$$

- Fourier decomposition appropriate for the *real* field:

$$\begin{aligned}\phi(t,\mathbf{x}) &= \phi_T(t) + \delta\phi(t,\mathbf{x}) \\ &= \phi_T(t) + V^{-1/2}\sum_{\mathbf{k}\neq 0}\left(U^*_{\mathbf{k}}(t)\mathcal{A}_{\mathbf{k}}(t) + U_{-\mathbf{k}}(t)\mathcal{A}^*_{-\mathbf{k}}(t)\right)e^{i\mathbf{k}\cdot\mathbf{x}},\end{aligned} \tag{37}$$

where $U_{\mathbf{k}}(t)$ is the periodic function (period $T/2$) of the canonical transformation.

- The next piece of the Hamiltonian is cubic in the fluctuations:

$$H_3 = \frac{\lambda}{6}\phi_T \int_{\text{box}} \delta\phi(\boldsymbol{x}, t)^3 d\boldsymbol{x}, \tag{38}$$

while the quadratic Hamiltonian governing φ to lowest order is

$$H_2 = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \mathcal{A}^*_{\boldsymbol{k}}(t)\mathcal{A}_{\boldsymbol{k}}(t), \tag{39}$$

where $(\mathcal{A}_{\boldsymbol{k}}, i\mathcal{A}^*_{\boldsymbol{k}})$ are conjugate canonically.

- The (Hamiltonian) equation of motion for $\mathcal{A}$ reads

$$\begin{aligned}
\frac{d\mathcal{A}_{\boldsymbol{k}}}{dt} &= -i\omega_{\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k}} + \frac{\partial H_3}{i\,\partial\mathcal{A}^*_{\boldsymbol{k}}} \\
&= -i\omega_{\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k}} + \int_{\text{box}} \frac{\delta H_3}{\delta\delta\phi(\boldsymbol{x},t)} \frac{\partial\delta\phi(\boldsymbol{x},t)}{i\,\partial\mathcal{A}^*_{\boldsymbol{k}}} d\boldsymbol{x} \\
&= -i\omega_{\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k}} - \frac{i\lambda\phi_T}{2} U_{\boldsymbol{k}}(t)\; V^{-1/2} \int_{\text{box}} \delta\phi(\boldsymbol{x},t)^2 e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} \\
&= -i\omega_{\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k}} - \frac{i\lambda\phi_T U_{\boldsymbol{k}}}{2}\; V^{-1/2} \\
&\quad \times \sum_{\boldsymbol{k}_1\boldsymbol{k}_2} \left(U^*_{\boldsymbol{k}_1}\mathcal{A}_{\boldsymbol{k}_1} + U_{-\boldsymbol{k}_1}\mathcal{A}^*_{-\boldsymbol{k}_1}\right)\left(U^*_{\boldsymbol{k}_2}\mathcal{A}_{\boldsymbol{k}_2} + U_{-\boldsymbol{k}_2}\mathcal{A}^*_{-\boldsymbol{k}_2}\right)\delta^{k}_{12},
\end{aligned} \tag{40}$$

- It is useful to eliminate the first term by making the change

$$\widetilde{\mathcal{A}_{\boldsymbol{k}}} = \mathcal{A}_{\boldsymbol{k}} e^{i\omega_k t}, \qquad \text{and c. c.} \tag{41}$$

- The resulting equation has the same form without the first term, but with the replacements

$$\mathcal{A}_{\boldsymbol{k}} \to \widetilde{\mathcal{A}_{\boldsymbol{k}}},$$
$$U_k \to \widetilde{U}_k = U_k e^{i\omega_k t},$$

$$\frac{d\widetilde{\mathcal{A}_{\boldsymbol{k}}}}{dt} = -\frac{i\lambda\phi_T \widetilde{U}_k}{2} V^{-1/2} \sum_{\boldsymbol{k}_1 \boldsymbol{k}_2} \left(\widetilde{U}^*_{\boldsymbol{k}_1} \widetilde{\mathcal{A}}_{\boldsymbol{k}_1} + \widetilde{U}_{-\boldsymbol{k}_1} \widetilde{\mathcal{A}}^*_{-\boldsymbol{k}_1} \right) \left(\widetilde{U}^*_{\boldsymbol{k}_2} \widetilde{\mathcal{A}}_{\boldsymbol{k}_2} + \widetilde{U}_{-\boldsymbol{k}_2} \widetilde{\mathcal{A}}^*_{-\boldsymbol{k}_2} \right) \delta^k_{12}. \tag{42}$$

- We see that the coupling constant λ serves to count the different orders in the weakly nonlinear expansion. Since $\lambda\Phi_T$ is $O(\lambda^{1/2})$, the perturbative expansion is a power series of $\lambda^{1/2}$. Therefore we use a formal expansion

$$\mathcal{A}_{\boldsymbol{k}}(t) = \mathcal{A}^{(0)}_{\boldsymbol{k}} + \mathcal{A}^{(1)}_{\boldsymbol{k}} + \mathcal{A}^{(2)}_{\boldsymbol{k}} + \dots, \qquad \mathcal{A}^{(i)}_{\boldsymbol{k}} \text{ is } O(\lambda^{i/2}), \tag{43}$$

and we will integrate term by term.

- We are interested in the evolution of the two-point correlation

$$\langle \mathcal{A}_{\boldsymbol{k}}(t)\mathcal{A}^*_{\boldsymbol{k}}(t)\rangle = \frac{V}{(2\pi)^d} n_k,$$

where n_k is called "wave action" or "particle number" distribution.

- The ensemble average is performed over the random phases θ and the random amplitudes J (see Nazarenko), where $\mathcal{A}_k = \sqrt{J_k}\theta_k$.
- With the aid of the Wick's contraction rule, one performs the phase averaging

$$\langle \theta_1^{(0)}\theta_2^{(0)}\theta_3^{(0)^*}\theta_4^{(0)*}\rangle_\theta = \delta_3^1\delta_4^2 + \delta_4^1\delta_3^2 - \delta_2^1\delta_3^1\delta_4^1,$$

where the upper index refers to the linear approximation, in which $\mathcal{A}_{\boldsymbol{k}}(t)$ is time-independent, $\mathcal{A}_{\boldsymbol{k}}^{(0)} = \mathcal{A}_{\boldsymbol{k}}(0)$.

- The averaging over random amplitudes is made by using

$$\langle J_1^{(0)} J_2^{(0)} \rangle = \begin{cases} \left\langle J_1^{(0)} \right\rangle \left\langle J_2^{(0)} \right\rangle, & \text{if } \boldsymbol{k}_1 \neq \boldsymbol{k}_2, \\ \left\langle \left(J_1^{(0)} \right)^2 \right\rangle, & \text{if } \boldsymbol{k}_1 = \boldsymbol{k}_2. \end{cases}$$

- The final expression for the fourth-order correlator of a random phase and amplitude field is

$$\left\langle \mathcal{A}_1^{(0)} \mathcal{A}_2^{(0)} \mathcal{A}_3^{(0)*} \mathcal{A}_4^{(0)*} \right\rangle = \left\langle J_1^{(0)} \right\rangle \left\langle J_2^{(0)} \right\rangle \left(\delta_3^1 \delta_4^2 + \delta_4^1 \delta_3^2 \right) + \left(\left\langle \left(J_1^{(0)} \right)^2 \right\rangle - 2 \left\langle J_1^{(0)} \right\rangle^2 \right) \delta_2^1 \delta_3^1 \delta_4^1 .$$

The second term is called the fourth-order cumulant. It vanishes for Gaussian fields.

- Perturbative expansion:

$$\mathcal{A}_{\boldsymbol{k}}(t)\mathcal{A}^*_{\boldsymbol{k}}(t) = \mathcal{A}^{(0)}_{\boldsymbol{k}}\mathcal{A}^{(0)*}_{\boldsymbol{k}} + \left(\mathcal{A}^{(0)}_{\boldsymbol{k}}\mathcal{A}^{(1)*}_{\boldsymbol{k}} + \text{c.c.}\right) + \left(\mathcal{A}^{(0)}_{\boldsymbol{k}}\mathcal{A}^{(2)*}_{\boldsymbol{k}} + \text{c.c.} + \mathcal{A}^{(1)}_{\boldsymbol{k}}\mathcal{A}^{(1)*}_{\boldsymbol{k}}\right) + O(\lambda^{3/2}),$$

- We define the functions

$$E(t,s,k,k_1,k_2) = \phi_T(t)\phi_T(s)\widetilde{U}_k(t)\widetilde{U}^*_k(s) \times \left[\widetilde{U}_{k_1}(t)\widetilde{U}^*_{k_1}(s) - \text{c.c.}\right]\left[\widetilde{U}_{k_2}(t)\widetilde{U}^*_{k_2}(s) + \text{c.c.}\right],$$

$$G(t,s,k,k_1,k_2) = \phi_T(t)\phi_T(s)\left[\widetilde{U}_k(t)\widetilde{U}^*_k(s) + \text{c.c.}\right] \times \left[\widetilde{U}_{k_1}(t)\widetilde{U}^*_{k_1}(s) + \text{c.c.}\right]\left[\widetilde{U}_{k_2}(t)\widetilde{U}^*_{k_2}(s) + \text{c.c.}\right].$$

- By using $\widetilde{U}_{\boldsymbol{k}}(t) = \widetilde{U}_{-\boldsymbol{k}}(t) = \widetilde{U}_k(t)$ and $J(\boldsymbol{k}) = J(-\boldsymbol{k})$, one obtains the following results:
  The average $\langle \mathcal{A}_{\boldsymbol{k}}^{(0)} \mathcal{A}_{\boldsymbol{k}}^{(1)*} \rangle$ vanishes.
  For $\boldsymbol{k} \neq 0$ the other products read:

$$\langle \mathcal{A}_{\boldsymbol{k}}^{(0)} \mathcal{A}_{\boldsymbol{k}}^{(2)*}(t) + \text{c.c.} \rangle = \lambda^2 V^{-1} \sum_{12} \delta_{12}^{k} \langle J_k^{(0)} \rangle \langle J_2^{(0)} \rangle \times \int_0^t dt' \int_0^{t'} ds \, 2\mathrm{Re} E(t', s, k, k_1, k_2),$$

$$\langle \mathcal{A}_{\boldsymbol{k}}^{(1)}(t) \mathcal{A}_{\boldsymbol{k}}^{(1)*}(t) \rangle = \frac{\lambda^2}{4} V^{-1} \sum_{12} \delta_{12}^{k} \langle J_1^{(0)} \rangle \langle J_2^{(0)} \rangle \times \int_0^t dt' \int_0^t ds \, G(t', s, k, k_1, k_2).$$

We need the asymptotics of the integrals when $T_1 \gg T$

$$\int_0^{T_1} dt \int_0^{T_1} ds\, G(t,s,k,k_1,k_2), \qquad \int_0^{T_1} dt \int_0^{t} ds\, 2\mathrm{Re}E(t,s,k,k_1,k_2).$$

We consider the infrared regime $k_i T \ll 1$. In such a regime

$$\omega_k = \frac{2\pi}{T} + \Delta\omega_k = \frac{2\pi}{T} + \frac{k}{\sqrt{3}} + O(k^3 T^3),$$

$$\tilde{U}_k(t) = \frac{T^2\sqrt{\lambda}}{2^5\, 3^{1/4}\, K(1/2)^2}\left(\frac{i}{\sqrt{k}}\dot{\phi}_T(t) - \frac{\sqrt{k}}{\sqrt{3}}\phi_T(t) + \dots\right) e^{i\Delta\omega_k t}$$

The replacement in the expression of G produces terms of the form

$$\frac{2C^6}{kk_1k_2}\cos\left[(s-t)(\Delta\omega_k + s_1\Delta\omega_{k_1} + s_2\Delta\omega_{k_2})\right]\phi_T(t)\dot{\phi}_T^3(t)\phi_T(s)\dot{\phi}_T^3(s), \tag{44}$$

and

$$C' kk_1k_2\cos\left[(s-t)(\Delta\omega_k + s_1\Delta\omega_{k_1} + s_2\Delta\omega_{k_2})\right]\phi_T(t)^4\phi_T(s)^4, \tag{45}$$

where $s_i = \pm 1$.

- The deltas at long time arise from

$$\frac{\sin^2\left(\frac{1}{2}xT_1\right)}{x^2} \to \frac{\pi T_1}{2}\delta(x), \qquad T_1 \to \infty.$$

- The terms proportional to $(kk_1k_2)^{-1}$ do not give rise to asymptotic closure because $\phi_T(t)\dot{\phi}_T^3(t)\phi_T(s)\dot{\phi}_T^3(s)$ has a zero mean value. The Fourier series does not have a constant term. The rule to be applied is in order to compute the leading behavior is

$$\phi_0(t)\dot{\phi}_0^3(t)\phi_0(s)\dot{\phi}_0^3(s) \to \frac{536\,870\,912\,\pi^6 q^2}{25(1-q^2)^2 T^{14}\lambda^4}\left(\pi^4 - 72K(1/2)^4\right)^2 \times \cos\left[\frac{4\pi(s-t)}{T}\right].$$

Therefore, the asymptotics is governed by

$$\delta\left(\frac{16\pi^2}{T^2} - (\Delta\omega_k + s_1\Delta\omega_{k_1} + s_2\Delta\omega_{k_2})^2\right).$$

Clearly, it is not possible to satisfy these constraints in the infrared regime.

- By contrast, the terms proportional to (kk_1k_2) produce asymptotic closure. Now the rule is

$$\phi_T(t)^4\phi_T(s)^4 \to \left(\frac{3072K^4}{T^4\lambda^2}\right)^2,$$

since the corresponding mean value of ϕ_T^4 is non zero. This gives rise to terms as

$$T_1\, kk_1k_2\delta\Big(\Delta\omega_k - \Delta\omega_{k_1} - \Delta\omega_{k_2}\Big).$$

The results are

$$\int_0^{T_1} dt \int_0^{T_1} ds\, G(t,s,k,k_1,k_2) = T_1 \frac{\pi T^4}{2304\sqrt{3}\,\lambda K^4} k k_1 k_2 \times \left(\delta(\Delta\omega_k - \Delta\omega_{k_1} - \Delta\omega_{k_2}) + \delta(\Delta\omega_k + \Delta\omega_{k_1} - \Delta\omega_{k_2}) + \delta(\Delta\omega_k - \Delta\omega_{k_1} + \Delta\omega_{k_2})\right),$$

$$\int_0^{T_1} dt \int_0^{t} ds\, 2\mathrm{Re}E(t,s,k,k_1,k_2) = T_1 \frac{\pi T^4}{4608\sqrt{3}\,\lambda K^4} k k_1 k_2 \times \left(-\delta(\Delta\omega_k - \Delta\omega_{k_1} - \Delta\omega_{k_2}) + \delta(\Delta\omega_k + \Delta\omega_{k_1} - \Delta\omega_{k_2}) - \delta(\Delta\omega_k - \Delta\omega_{k_1} + \Delta\omega_{k_2})\right), \quad T_1 \to \infty.$$

Kinetic equation

- We now set

$$T \ll T_1 \ll \frac{T}{\lambda} \Rightarrow \frac{F(T_1) - F(0)}{\lambda T_1} \approx \frac{1}{\lambda} \frac{\partial F}{\partial T_1}.$$

This replacement involves the filtering-out of rapid oscillations on the time scale of order T. (See Nazarenko. Other approach based in multiple time scales has been used by Newell 1968).

- Finally the kinetic equation emerges in the following form

$$\partial_t n_k = \int (\mathcal{R}_{12k} - \mathcal{R}_{k12} - \mathcal{R}_{2k1}) d\boldsymbol{k}_1 d\boldsymbol{k}_2,$$

where

$$\mathcal{R}_{12k} = \frac{\text{cons}}{\rho_T} k_1 k_2 k\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \delta(\Delta\omega_k - \Delta\omega_{k_1} - \Delta\omega_{k_2}) \\ \times (n_1 n_2 - n_2 n_k - n_k n_1),$$

and $\rho_T \propto \lambda^{-1} T^{-4}$ is the energy density of the periodic solution.

- A perturbative expansion around homogeneous periodic solutions of $\lambda\phi^4$ model is feasible. The natural parameter of expansion is $\lambda^{1/2}$.
- The resulting kinetic equation has the same form as that appears in the nonlinear Schrödinger model. It is found that an asymptotic closure with a three-wave coupling is possible, and the linear dispersion relation is $\Delta\omega_k \propto k$. The role of the condensate is played here by the energy density $\rho \propto \lambda^{-1} T^{-4}$.
- The kinetic equation seems to describe the interaction of Goldstone modes from a broken global symmetry. The pattern of breaking must correspond to $\mathbb{R} \to \mathbb{Z}$, while in the NLS model it was $U(1) \to 1$.
- For completeness, the energetics of the whole solution $\phi(t, \boldsymbol{x}) = \phi_T(t) + \delta\varphi(t, \boldsymbol{x})$ must be studied. Specifically, it is expected that T becomes time-dependent over some slow time scale (e. g. T/λ), in order to compensate the energy stored in the inhomogeneities.
- The effects of parametric instability do not have been discussed. They deserve further study.

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