Finite Elements for the Quasi-Geostrophic Equations of the Ocean

Erich L Foster

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In collaboration with: Traian Iliescu (VT), Zhu Wang (IMA), and Dave Wells (VT)

- National Science Foundation Grant DMS-1025314
- Institute for Critical Technology and Applied Science (ICTAS) fund 118709
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Large Scale Ocean Surface Currents

Quasi-Geostrophic Equations

Argyris Finite Element

Optimal Error Estimates

Time Dependence

Future Work
1. Large Scale Ocean Surface Currents

2. Quasi-Geostrophic Equations

3. Argyris Finite Element

4. Optimal Error Estimates

5. Time Dependence

6. Future Work
1 Large Scale Ocean Surface Currents

2 Quasi-Geostrophic Equations

3 Argyris Finite Element

4 Optimal Error Estimates

5 Time Dependence

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The large scale surface currents of the ocean, Haidvogel 1999.
Characteristics of large scale oceanic surface currents

- Driven by forces such as
  - Wind
  - Coriolis Force, i.e. the deflection of moving objects due to the rotation of the Earth.
- Large scale gyres
- Strong western boundary currents
- Scales

\[ L = O(10^6 \text{m}) \quad U = O(10^{-2} \text{m/s}) \]
\[ T = \frac{L}{U} = O(3 \text{yr}) \quad D = O(10^3 \text{m}) \]

Vallis 2006
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Vallis 2006
How do we model the Large Scale Currents?
Modeling Large Scale Currents

- Assumptions and Simplifications
  - Ocean width is much larger than the depth, i.e. \( L \gg D \).
  - The Coriolis force varies only in the \( y \)-direction.
  - Quasi-Geostrophic balance, i.e. the pressure gradient force is nearly balanced by the Coriolis force.

- Streamfunction
  \[
  u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}
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Quasi-Geostrophic Equations

- The QGE are a simplified model for planet-scale flows.

Streamfunction-Vorticity Formulation

\[
\frac{\partial q}{\partial t} + J(\psi, q) = -Re^{-1} \Delta q + F \tag{1}
\]

\[
q = -Ro \Delta \psi + y \tag{2}
\]

where \( Ro, Re \) are the Rossby and Reynolds numbers, and the Jacobian

\[
J(\xi, \eta) = \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x}
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- The Jacobian is associated with \((u \cdot \nabla) u\) in the Navier-Stokes Equations
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The Rossby number is the ratio of inertial and Coriolis forces

\[ Ro = \frac{U}{\beta L^2} \]

- Low Rossby number, rotation is important
- High Rossby number, rotation is not important

The Reynolds number is the ratio of the inertial and viscous forces

\[ Re = \frac{UL}{A} \]

- High Reynolds number, inertial forces dominate
- Low Reynolds number, viscous forces dominate

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Brief History of the QGE

(Courtesy of Peter Lynch, University college, Dublin)

- Developed by Charney in 1948
- The first one day forecast was made in April 1950, on ENIAC.
- Von Neumann, Charney, Fjørtoft, Smagorinksy, and more.
- Took longer than 24 hours to predict the weather 24 hours in the future.
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\[ \frac{\partial q}{\partial t} + J(\psi, q) = -Re^{-1} \Delta q + F \]

\[ q = -Ro \Delta \psi + y \]

---

**Pure Streamfunction Formulation**

\[ -\frac{\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F \]  

(3)

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<tr>
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QGE is not Navier-Stokes

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\begin{align*}
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\text{QGE} & \quad \frac{\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F
\end{align*}
\]

\begin{align*}
\text{NSE} & \quad Ro = 1 \\
\text{QGE} & \quad Ro = 0.1 \\
\text{QGE} & \quad Ro = 0.01 \\
\text{QGE} & \quad Ro = 0.001
\end{align*}

Time Averaged, \( t = [0, 10], \ dt = 1 \times 10^{-3}, \ Re = 200, \ F = \sin \pi y \)
Stationary Quasi-Geostrophic Equations

- Pure Streamfunction Form of Stationary QGE

\[
Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F
\]

\[
\psi = 0, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega
\]

- Weak Form

Find \( \psi \in X \) such that \( \forall \chi \in X \)

\[
Re^{-1} (\Delta \psi, \Delta \chi) + b(\psi; \psi, \chi) - Ro^{-1} (\psi_x, \chi) = Ro^{-1} (F, \chi)
\]

\[
b(\zeta; \xi, \eta) = [(\Delta \zeta \cdot \xi_y, \eta_x) - (\Delta \zeta \cdot \xi_x, \eta_y)]
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Weak Formulation

**Space**

\[ X = H^2_0(\Omega) = \left\{ \psi \in H^2(\Omega) \mid \psi = 0, \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega \right\} \]

**Bounds on forms**

\[ (\Delta \psi, \Delta \chi) \leq |\psi|_2 |\chi|_2 \quad \forall \psi, \chi \in X \]
\[ b(\zeta; \psi, \chi) \leq \Gamma_1 |\zeta|_2 |\psi|_2 |\chi|_2 \quad \forall \zeta, \psi, \chi \in X \]
\[ (\psi_x, \chi) \leq \Gamma_2 |\psi|_2 |\chi|_2 \quad \forall \psi, \chi \in X \]
\[ (F, \chi) \leq \|F\|_{-2} |\chi|_2 \quad \forall \chi \in X \]

**Stability bound**

\[ |\psi|_2 \leq \text{Re} \, Ro^{-1} \|F\|_{-2} \]
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- Stability bound

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Finite Element Formulation

- Conforming Finite Element Space

\[ X^h \subset X = \left\{ \psi^h \in H^2(\Omega) \mid \psi^h = 0, \frac{\partial \psi^h}{\partial n} = 0 \text{ on } \partial \Omega \right\} \]

- FE Form

Find \( \psi^h \in X^h \) such that \( \forall \chi^h \in X^h \)

\[
Re^{-1}(\Delta \psi^h, \Delta \chi^h) + b(\psi^h; \psi^h, \chi^h) - Ro^{-1}(\psi^h_x, \chi^h) = Ro^{-1}(F, \chi^h)
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(6)

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Examples of $C^1$ Finite Elements

- Bogner-Fox-Schmit Rectangle

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The Argyris Finite Element is $C^1$

- Conforming Finite Element for fourth-order problems
- Fifth-order basis functions
- Interpolation Error Bounds for Argyris

$$\|u - P^h u\|_p \leq C h^{4+\sigma} |u|_p \quad \text{for} \quad \sigma = 0, 1, 2$$

- 21 degrees of freedom
  - Function values at each vertex (3 values total)
  - First derivative values at each vertex (6 values total)
  - Second derivative values at each vertex (9 values total)
  - Normal derivative values at the midpoints (3 values total)

- https://github.com/VT-ICAM/ArgyrisPack
The Argyris Finite Element is $C^1$

- Conforming Finite Element for fourth-order problems
- Fifth-order basis functions
  - Interpolation Error Bounds for Argyris
    \[
    \|u - I^h u\|_{2-s} \leq C h^{4+s} |u|_6 \text{ for } s = 0, 1, 2
    \] (7)
- 21 degrees of freedom
  - Function values at each vertex (3 values total)
  - First derivative values at each vertex (6 values total)
  - Second derivative values at each vertex (9 values total)
  - Normal derivative values at the midpoints (3 values total)

https://github.com/VT-ICAM/ArgyrisPack
The Argyris Finite Element is $C^1$
- Conforming Finite Element for fourth-order problems
- Fifth-order basis functions
  - Interpolation Error Bounds for Argyris
    \[ \|u - I^h u\|_{2-s} \leq C h^{4+s} |u|_6 \text{ for } s = 0, 1, 2 \] (7)
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First Derivative Values of Argyris Triangle

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  - Conforming Finite Element for fourth-order problems
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- https://github.com/VT-ICAM/ArgyrisPack
Pros/Cons

- **Cons:**
  - 21 Degrees of Freedom
  - Normal Derivatives

- **Pros:**
  - High Order of Convergence
  - $C^1$

Normal derivatives aren’t respected by the affine transformation

\[
F(\hat{x}) = B\hat{x} + x_1 = \begin{bmatrix}
  x_2 - x_1 & x_3 - x_1 \\
  y_2 - y_1 & y_3 - y_1
\end{bmatrix} \begin{bmatrix}
  \hat{x} \\
  \hat{y}
\end{bmatrix} + \begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\]

- Need a transformation $C$ that will take us from the reference triangle $\hat{K}$ to any general triangle $K$. 
Pros/Cons

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- Need a transformation $C$ that will take us from the reference triangle $\hat{K}$ to any general triangle $K$. 

E. L. Foster (BCAM)
Transformation?

Reference triangle, $\hat{K}$, to general triangle, $K$?

$C = \hat{DE}$, (8)

from V. Dominguez and F. J. Sayas 2006.
Transformation?

Reference triangle, $\hat{K}$, to general triangle, $K$?

$$C = \hat{D}E,$$

from V. Dominguez and F. J. Sayas 2006.
First some notation.

\[ p_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \]

\[ p_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \]

\[ p_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \]

\[ n_2 = \frac{1}{\ell_2} R \cdot v_2 \]

\[ v_2 = p_3 - p_1 \]

\[ v_1 = p_3 - p_2 \]

\[ n_3 = \frac{1}{\ell_3} R \cdot v_3 \]

\[ n_1 = \frac{1}{\ell_1} R \cdot v_1 \]

\[ v_3 = p_2 - p_1 \]

Notation corresponding to the Dominguez transformation.
What to do about the derivatives?

- Relationship between $\nabla_x$ and $\nabla_{\hat{x}}$

$$\nabla_{\hat{x}}(\varphi \circ F) = B^T \nabla_x(\varphi) \circ F$$

- Relationship between $H_x = [\partial_{xx}, \partial_{xy}, \partial_{yy}]^T$ and $H_{\hat{x}} = [\partial_{\hat{x}x}, \partial_{\hat{x}y}, \partial_{\hat{y}y}]^T$

$$H_{\hat{x}}(\varphi \circ F) = \Theta H_x(\varphi) \circ F,$$

where

$$\Theta = \begin{bmatrix}
B_{11}^2 & 2B_{11}B_{21} & B_{21}^2 \\
B_{12}B_{11} & B_{12}B_{21} + B_{11}B_{22} & B_{21}B_{22} \\
B_{12}^2 & 2B_{22}B_{12} & B_{22}^2
\end{bmatrix}$$
What to do about the derivatives?

- Relationship between $\nabla_x$ and $\nabla_{\tilde{x}}$

$$\nabla_{\tilde{x}}(\varphi \circ F) = B^T \nabla_x(\varphi) \circ F$$

- Relationship between $H_x = [\partial_{xx}, \partial_{xy}, \partial_{yy}]^T$ and $H_{\tilde{x}} = [\partial_{\tilde{xx}}, \partial_{\tilde{xy}}, \partial_{\tilde{yy}}]^T$

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B_{12}^2 & 2B_{22}B_{12} & B_{22}^2
\end{bmatrix}$$
What about the normal derivatives?

- Rotation matrix (CCW 90°)

\[
R = \begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}
\]

\[
f_i = \frac{1}{\ell_i^2 |\hat{v}_i|} R \hat{v}_i \cdot B^T R v_i,
\]
\[
g_i = \frac{1}{\ell_i^2 |\hat{v}_i|} R \hat{v}_i \cdot B^T v_i
\]

- Let

\[
Q = \begin{bmatrix}
f_1 & f_2 & f_3 & g_1 & g_2 & g_3 \\
\end{bmatrix}
\]

\[
D = \text{diag}[I_3, B^T, B^T, B^T, \Theta, \Theta, \Theta, Q]
\]
What about the normal derivatives?

- Rotation matrix (CCW 90°)

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R = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

\[
f_i = \frac{1}{\ell_i^2 |\hat{v}_i|} R\hat{v}_i \cdot B^T R v_i, \quad g_i = \frac{1}{\ell_i^2 |\hat{v}_i|} R\hat{v}_i \cdot B^T v_i
\]

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\]

- **Let**

\[
Q = \begin{bmatrix}
f_1 & g_1 \\
f_2 & g_2 \\
f_3 & g_3
\end{bmatrix}
\]

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What about the normal derivatives?

- Rotation matrix (CCW 90°)

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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$$f_i = \frac{1}{\ell^2_i |\hat{v}_i|} R\hat{v}_i \cdot B^T Rv_i, \quad g_i = \frac{1}{\ell^2_i |\hat{v}_i|} R\hat{v}_i \cdot B^T v_i$$

- Let

$$Q = \begin{bmatrix} f_1 & | & g_1 \\ f_2 & | & g_2 \\ f_3 & | & g_3 \end{bmatrix}$$

$$D = \text{diag}[I_3, B^T, B^T, B^T, \Theta, \Theta, \Theta, Q]$$
\[ E = \begin{bmatrix} I_{18} & 0 \\ 0 & L \\ T & 0 \end{bmatrix} \quad L = \text{diag}[\ell_1, \ell_2, \ell_3] \]

\[ T \text{ is composed of the three sub blocks} \]

\[ \frac{15}{8} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad -\frac{7}{16} \begin{bmatrix} v_1^T & v_1^T & 0 \\ v_2^T & 0 & v_2^T \\ 0 & v_3^2 & v_3^T \end{bmatrix}, \]

\[ \frac{1}{32} \begin{bmatrix} -w_1^T & w_1^T & 0 \\ -w_2^T & 0 & w_2^T \\ 0 & -w_3^2 & w_3^T \end{bmatrix} \]

where \( w_i^T = [(v_i^x)^2, 2v_i^x v_i^y, (v_i^y)^2] \).

https://github.com/VT-ICAM/ArgyrisPack
Pitfalls
Caveat to always rotating CCW $90^\circ$

Discretized Domain
Caveat to always rotating CCW $90^\circ$
Caveat to always rotating CCW 90°
Caveat to always rotating CCW 90°
Caveat to always rotating CCW $90^\circ$

How to prevent this?

- Check node number and multiply by negative when appropriate.
- Number the nodes in a particular way to avoid the problem.
Boundary Conditions

- Consider the Poisson Problem

\[- \Delta u = f \text{ on } \Omega, \]
\[u = 0 \text{ on } \partial \Omega.\]

- With $C^0$ FEs we can set $u^h = 0$ on $\partial \Omega$.
Boundary Conditions

- Consider the Poisson Problem

\[- \Delta u = f \text{ on } \Omega,\]
\[u = 0 \text{ on } \partial \Omega.\]

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Boundary Conditions

- Consider the Poisson Problem

\[ -\Delta u = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega. \]

- With $C^0$ FE, we can set $u^h = 0$ on $\partial\Omega$

Eliminate Boundary Nodes
Boundary Conditions

- Consider the Poisson Problem

\[- \Delta u = f \text{ on } \Omega,\]
\[u = 0 \text{ on } \partial \Omega.\]

- With $C^0$ FEs we can set $u^h = 0$ on $\partial \Omega$

- Why can’t we do the same for $C^1$ elements?
  Well we can we just have to be smarter, since, for the Argyris element,

\[
\frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \frac{\partial}{\partial n} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial x \partial y} \frac{\partial^2}{\partial y^2}
\]

are all DoFs.
$\partial \Omega = \Gamma_1 \cup \Gamma_2$

Rectangular Problem Domain
Boundary Conditions for Rectangular Domain
\[ u = 0 \quad \Gamma_1 \qquad \Omega \qquad \Gamma_1 \quad u = 0 \]

\[ \Rightarrow u = \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0 \]

**\( \Gamma_1 \) Boundary Conditions**

We would have to set

\[ u^h = \frac{\partial u^h}{\partial y} = \frac{\partial^2 u^h}{\partial y^2} = 0 \text{ on } \Gamma_1 \]

for the Argyris Element.
\[ u = 0 \]

\[ \Gamma_2 \]

\[ \Omega \]

\[ \Gamma_2 \]

\[ u = 0 \]

\[ \Gamma_2 \text{ Boundary Conditions} \]
$u = 0$

$\Omega$

$\Gamma_2$

$u = 0$

$\Rightarrow u = \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = 0$

$\Gamma_2$ Boundary Conditions

We would have to set

$$u^h = \frac{\partial u^h}{\partial x} = \frac{\partial^2 u^h}{\partial x^2} = 0 \text{ on } \Gamma_2$$

for the Argyris Element.
But what if we change the domain?

- Analyze the BCs and implement accordingly

- Lagrange Multipliers
  - Let $\Lambda$ be the matrix representation of constraints (BCs)
  - Then the original Poisson problem can be written as a FE system with Lagrange multipliers:

\[
K u^h + \Lambda^T \lambda = \ell, \\
\Lambda u^h + \varepsilon \lambda = 0,
\]

where $\lambda$ is the Lagrange multipliers, $K$ is the stiffness matrix, $\varepsilon$ is a small perturbation to prevent a poorly conditioned system, and $\ell$ is the load vector.
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where $\lambda$ is the Lagrange multipliers, $K$ is the stiffness matrix, $\varepsilon$ is a small perturbation to prevent a poorly conditioned system, and $\ell$ is the load vector.
Recall the SQGE

\[ Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F \]

\[ \psi = 0, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega \]

**Theorem (Optimal Error Estimate)**

Let \( \psi \) be a unique solution of (5) and \( \psi^h \) be the solution of (6). Furthermore, assume that the following small data condition is satisfied:

\[ Re^{-2} Ro \geq \Gamma_1 \| F \|_{-2} \]

Then

\[ |\psi - \psi^h|_2 \leq c(Re, Ro, \Gamma_1, \Gamma_2, F) \cdot \inf_{\psi^h \in X^h} |\psi - \psi^h|_2 \quad (9) \]

where

\[ c(Re, Ro, \Gamma_1, \Gamma_2, F) := \frac{\Gamma_2 Ro^{-1} + 2Re^{-1} + \Gamma_1 Re Ro^{-1} \| F \|_{-2}}{Re^{-1} - \Gamma_1 Re Ro^{-1} \| F \|_{-2}} \]
Recall the SQGE

\[ Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F \]

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Theorem (Argyris Error Estimates)

Let \( \psi \) be the solution of (5) and \( \psi^h \) be the solution of (6) and assume the small data condition

\[
Re^{-2} Ro \geq \Gamma_1 \| F \|_{-2}.
\]

Furthermore, assume that \( \psi \in H^6(\Omega) \cap H_0^2(\Omega) \) Then there exists positive constants \( C_0, C_1, \) and \( C_2 \) that depend on \( Re, Ro, \Gamma_1, \Gamma_2, F \) but not \( h \) such that

\[
|\psi - \psi^h|_2 \leq C_2 \cdot h^4 \tag{10}
\]
\[
|\psi - \psi^h|_1 \leq C_1 \cdot h^5 \tag{11}
\]
\[
\|\psi - \psi^h\|_0 \leq C_0 \cdot h^6 \tag{12}
\]


Proof relies on duality argument (Aubin-Nitsche) to bootstrap from \( H^2 \)-norm into \( L^2 \)-norm.
Numerical Test

\[ Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F \]

\[ \psi = 0, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega, \quad Re = Ro = 1 \]

- Domain

\[ \Omega = [0, 1] \times [0, 1] \]

- Exact Solution

\[ \psi(x, y) = \sin^2 \pi x \cdot \sin^2 2\pi y \]
Numerical Test

\[ Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F \]

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<table>
<thead>
<tr>
<th>( h )</th>
<th>DoFs</th>
<th>( e_0 )</th>
<th>( L_2 ) order</th>
<th>( e_1 )</th>
<th>( H^1 ) order</th>
<th>( e_2 )</th>
<th>( H^2 ) order</th>
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<td>0.009499</td>
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<td>6.182</td>
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<td>5.099</td>
<td>0.0001191</td>
<td>4.025</td>
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</table>
The Mediterranean Sea

- Domain:

Mesh for the Mediterranean Sea with $DoFs = 240, 342$.

- $Re = 5.27$, $Ro = 6.051 \times 10^{-4}$, $F = \sin \frac{\pi}{4} y$.
- Take the “true” solution to be the solution obtained from the numerical simulation on the fine mesh, $DoFs = 955, 302$ or $h = \frac{1}{640}$. 

“True” solution of SQGE applied to the Mediterranean Sea.

Observed surface currents of the Mediterranean Sea.
Rates of Convergence

<table>
<thead>
<tr>
<th>$h$</th>
<th>$DoFs$</th>
<th>$e_0$</th>
<th>$L^2$ order</th>
<th>$e_1$</th>
<th>$H^1$ order</th>
<th>$e_2$</th>
<th>$H^2$ order</th>
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<td>$1.49 \times 10^{-2}$</td>
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<tr>
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<td>240,342</td>
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<td>1.72</td>
<td>$4.35 \times 10^{-3}$</td>
<td>0.994</td>
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Time Dependence

- Recall the pure streamfunction form of QGE, i.e. Equation 3
  $$\frac{-\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F$$

$$\psi(t; x, y) = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega, \quad \psi(0; x, y) = \psi_0(x, y)$$

- Strong Form
  $$\langle \nabla \psi_t, \nabla \chi \rangle + Re^{-1} \langle \Delta \psi, \Delta \chi \rangle + b(\psi; \psi, \chi)$$
  $$- Ro^{-1} \langle \psi_x, \chi \rangle = Ro^{-1} \langle F, \chi \rangle, \quad \forall \chi \in X \quad (13)$$

- Semi-discretization
  $$\langle \nabla \psi_t^h, \nabla \chi^h \rangle + Re^{-1} \langle \Delta \psi^h, \Delta \chi^h \rangle + b(\psi^h; \psi^h, \chi^h)$$
  $$- Ro^{-1} \langle \psi_x^h, \chi^h \rangle = Ro^{-1} \langle F, \chi^h \rangle, \quad \forall \chi^h \in X^h \quad (14)$$
Time Dependence

- Recall the pure streamfunction form of QGE, i.e. Equation 3

\[
- \frac{\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F
\]

\[
\psi(t; x, y) = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega, \quad \psi(0; x, y) = \psi_0(x, y)
\]

- Strong Form

\[
(\nabla \psi_t, \nabla \chi) + Re^{-1}(\Delta \psi, \Delta \chi) + b(\psi; \psi, \chi)
\]

\[
- Ro^{-1}(\psi_x, \chi) = Ro^{-1}(F, \chi), \quad \forall \chi \in X
\] (13)

- Semi-discretization

\[
(\nabla \psi_t^h, \nabla \chi^h) + Re^{-1}(\Delta \psi^h, \Delta \chi^h) + b(\psi^h; \psi^h, \chi^h)
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- Strong Form

\[(\nabla \psi_{t}, \nabla \chi) + Re^{-1}(\Delta \psi, \Delta \chi) + b(\psi; \psi, \chi) - Ro^{-1}(\psi_{x}, \chi) = Ro^{-1}(F, \chi), \quad \forall \chi \in X \quad (13)\]

- Semi-discretization

\[(\nabla \psi_{t}^{h}, \nabla \chi^{h}) + Re^{-1}(\Delta \psi^{h}, \Delta \chi^{h}) + b(\psi^{h}; \psi^{h}, \chi^{h}) - Ro^{-1}(\psi_{x}^{h}, \chi^{h}) = Ro^{-1}(F, \chi^{h}), \quad \forall \chi^{h} \in X^{h} \quad (14)\]
Theorem (Semi-Discretization)

Let $\psi$ be a unique solution to the QGE, (3). Then there exists constants $C_1(T, Re, \Gamma_3)$, $C_2(T, Re, \Gamma_3)$, $C_3(Re, Ro, \Gamma_2)$, and $C_4(T, F, \psi_0, \Gamma_1, \Gamma_2, Re, Ro, \|\Delta\psi\|_{L^4(0,T;L^2)})$ such that for all $t \in [0, T]$

$$
\|\nabla(\psi - \psi^h)\|^2 + Re^{-1} \int_0^T \|\Delta(\psi - \psi^h)\|^2 dt \\
\leq C_1(T, Re, \Gamma_3) \|\nabla(\psi_0 - \psi^h(0))\|^2 \\
+ \inf_{\lambda^h(t) \in X^h} \left\{ C_2(T, Re, \Gamma_3) \int_0^T \|\nabla(\psi - \lambda^h)_{t}\|^2_{-1} \\
+ C_3(Re, Ro, \Gamma_2) \|\Delta(\psi - \lambda^h)\|^2 dt \\
+ C_4(T, Re, Ro, F, \Gamma_1, \Gamma_2, \Gamma_4, \|\Delta\psi\|_{L^4(0,T;L^2)}) \\
\|\Delta(\psi - \lambda^h)\|_{L^4(0,T;L^2)} + 2 \|\nabla(\psi - \lambda^h)\|^2 \right\},
$$

(15)
Theorem (QGE Rates of Convergence)

Let $X^h$ be the FE space associated with the Argyris element and an $I^h$ the associated $\mathbb{P}^5$-interpolation operator (see Theorem 6.1.1 in Ciarlet). Suppose the interpolation estimates (7) hold and that $\psi, \psi_t \in H^6(\Omega)$. Suppose also that the assumptions of 3 hold. Then,

$$\| \nabla \left( \psi - \psi^h \right) (T) \|^2 + Re^{-1} \int_0^T \| \Delta \left( \psi - \psi^h \right) \|^2 \, dt \\
\leq h^8 C \left\{ \left( C_1(T, Re, \Gamma_3) + 2 \right) h^2 |\psi|^2_6 + C_2(T, Re, \Gamma_3) \right. \\
\left. \left( h^2 \| \psi_t \|^2_{L^2(0,T;H^6(\Omega))} + C_3(Re, Ro, \Gamma_2) \| \psi \|^2_{L^2(0,T;H^6(\Omega))} \right) \\
+ C_4(T, F, \psi_0, Re, Ro, \Gamma_1, \Gamma_3, \Gamma_4, \| \psi \|_{L^4(0,T;L^2)} \| \psi \|^2_{L^4(0,T;H^6(\Omega))} \right\}.$$

(16)
Numerical Test

\[- \frac{\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F\]

\[\psi = 0, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial \Omega, \quad \psi(0; x, y) = \psi_0(x, y)\]

\[Re = Ro = 1\]

- \[\Omega = [0, 1] \times [0, 1], \quad t = [0, 0.5], \quad dt = \frac{1}{8192}\]
- \[\psi(t; x, y) = [\sin \pi x \sin \pi y]^2 \sin t\]
Numerical Test

\[-\frac{\partial [\Delta \psi]}{\partial t} + Re^{-1} \Delta^2 \psi + J(\psi, \Delta \psi) - Ro^{-1} \frac{\partial \psi}{\partial x} = Ro^{-1} F\]

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- \(\psi(t; x, y) = [\sin \pi x \sin \pi y]^2 \sin t\)

<table>
<thead>
<tr>
<th>(h)</th>
<th>DoFs</th>
<th>(e_{L^2})</th>
<th>(L^2) order</th>
<th>(e_{H^1})</th>
<th>(H^1) order</th>
<th>(e_{H^2})</th>
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<td>(1.23 \times 10^{-2})</td>
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<td>(1.18 \times 10^{-1})</td>
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<td>(3.43 \times 10^{-7})</td>
<td>4.12</td>
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</table>
Mesh of the North Atlantic created using GMSH with

\[ h = \frac{1}{100}, \quad \text{DoFs} = 79,635. \]
Mesh of the North Atlantic created using GMSH with 
\( h = \frac{1}{100}, \text{DoFs} = 79,635. \)

QGE on North Atlantic

\[ F = \sin(\pi y), \ Re = 5.27, \]
\[ Ro = 6.051 \times 10^{-4}, \ t = [0, 3], \]
\[ dt = \frac{1}{100}, \text{DoFs} = 79,635 \]
Remember QGE is not NSE.

NSE: $Re = 200$

QGE: $Re = 200, Ro = 1$

QGE: $Re = 200, Ro = 1 \times 10^{-1}$

QGE: $Re = 200, Ro = 1 \times 10^{-2}$

QGE: $Re = 200, Ro = 1 \times 10^{-3}$
Challenges and Future Work

- **Realistic Parameters and Domains**
  - **Challenge**
  - Narrow Boundary Layer
  - Dynamic Structures
  - Loss of Regularity
  - Islands
  - **Possible Solution**
  - Stabilization Methods
  - Adaptive Mesh Refinement
  - Radical Meshes
  - Integral Conditions as in van Gijzen 1998

- **Ensemble Forecasting**
  - **Challenge**
  - “Slow” Code
  - **Possible Solutions**
  - Parallel Processing
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  - Large Eddy Simulation

- **Realistic Forcing through data assimilation.**
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E. L. Foster (BCAM)
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- Quasi-Geostrophic Equation
- Pure Stream Function form of QGE
- Optimal Error estimates
- Numerical Confirmation of Error Estimates
- Future Work

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References

- C. Johnson, 2009.