The Alternating Descent Method: Numerical Tests for the Augmented Burgers Equation

Task D: Numerical methods for flow control in the presence of shocks
Subtask D.e: Possible use of these techniques for viscous systems

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Outline

Statement of the problem

We want to solve this optimization problem:

$$\mathcal{J}(u^0) = \min_{u^0 \in \mathcal{U}_{ad}} \frac{1}{2} \int_{\mathbb{R}} [u(x, T) - u^*(x)]^2 \, dx$$

subject to

$$\begin{cases} u_t + [f_{c,\theta}(u)]_x = \nu u_{xx} + \frac{c}{\theta^3} \left(K_\theta * u - \theta u \right), & (x,t) \in \mathbb{R} \times (0, T) \\ u(x, 0) = u(x), & x \in \mathbb{R} \end{cases}$$

where $f_{c,\theta}(u) = -\frac{u^2}{2} - \frac{c}{\theta} u$ and $K_\theta(z) := \chi_{z>0}(z) e^{-z/\theta}$.

Linearized equation

Let us consider

$$u^{\varepsilon,0}(x) := u^0(x + \varepsilon\delta\varphi^0(x)) + \varepsilon\delta u^0(x)$$

If $\varepsilon > 0$ is small enough, u^ε can be written as $u^\varepsilon = u + \varepsilon(\delta u + \delta\varphi u_x) + \mathcal{O}(\varepsilon^2)$, where:

$$\begin{cases} \partial_t \delta\phi + \partial_x \left[f'_{c,\theta}(u)\delta\phi \right] = \nu \partial_{xx}\delta\phi + \frac{c}{\theta^3}\left(K_\theta * \delta\phi - \theta\delta\phi \right), & (x,t) \in \mathbb{R} \times (0,T) \\ \delta u(x,0) = \delta u^0(x), & x \in \mathbb{R} \end{cases}$$

denoting $\delta\phi = \delta u + \delta\varphi u_x$.
In this case, the Gateaux derivative of $\mathcal{J}$ is:

$$\delta\mathcal{J}(u^0)[\delta\phi^0] = \lim_{\varepsilon\to 0} \frac{\mathcal{J}(u^{\varepsilon,0}) - \mathcal{J}(u^0)}{\varepsilon} = \int_{\mathbb{R}} \left[u(x,T) - u^*(x) \right] \delta\phi(x,T) dx$$

Adjoint equation

Multiply by a function $p \in \mathcal{C}_0^1(\mathbb{R} \times [0, T])$ and integrate it on $\mathbb{R} \times [0, T]$. Integrating by parts and rearranging terms:

$$\int_{\mathbb{R}} \delta\phi(x, T) p(x, T) dx - \int_{\mathbb{R}} \delta\phi(x, 0) p(x, 0) dx$$
$$+ \int_0^T \int_{\mathbb{R}} \delta\phi \left[-p_t - f'_{c,\theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3} \left(\bar{K}_\theta * p - p \right) \right] dxdt = 0$$

where $\bar{K}_\theta(z) := K_\theta(-z)$. This leads to the adjoint equation:

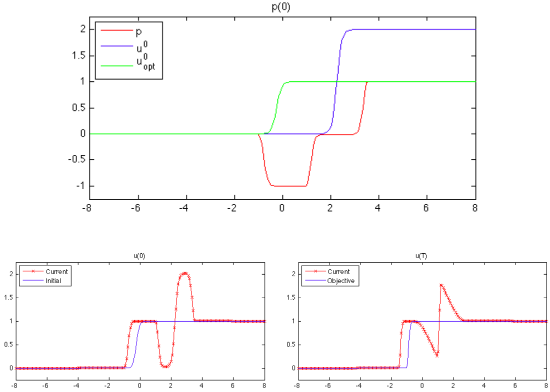
$$\begin{cases} -p_t - f'_{c,\theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3} \left(\bar{K}_\theta * p - \theta p \right) = 0, & (x, t) \in \mathbb{R} \times (0, T) \\ p(x, T) = u(x, T) - u^*(x), & x \in \mathbb{R} \end{cases}$$

and, therefore,

$$\delta\mathcal{J}(u^0)[\delta\phi^0] = \int_{\mathbb{R}} p(x, 0) \left[\delta u^0(x) + \delta\varphi^0(x) u_x^0(x) \right] dx$$

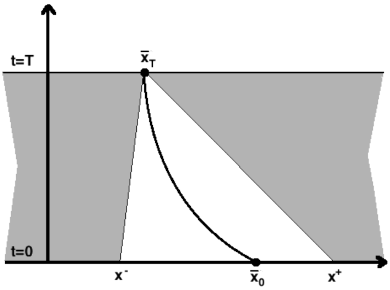
Usual continuous approach

Why the usual continuous approach (assuming u is smooth and $\delta\varphi^0 \equiv 0$) does not work well?



Alternating Descent Method

If we consider the viscosity to be small, the viscous case behaves similarly to the the inviscid one. Thus, from the numerical point of view, it seems natural to try to apply the same ideas.



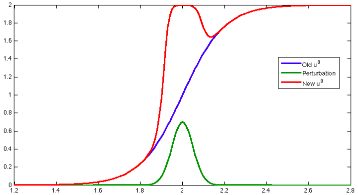
Our aim is to obtain two pairs $(\delta u^0, \delta\varphi^0)$, one acting mainly on $[x^-, x^+]$ and the other out of it.

Descent directions

A possible option is to choose $\delta u^0 \equiv 0$, so that

$$\delta \mathcal{J} = \int_{\mathbb{R}} p(x,0)\delta\varphi^0(x)u_x^0(x)dx \tag{1}$$

This defines the second term of the descent direction: $\delta\varphi^0 = -p(x,0)u_x^0(x)$. But this perturbation could develop undesirable extra oscillations or, even worse, extra quasi-shocks.



Descent directions

The second option is to choose $(\delta u^0, \delta\varphi^0)$ such that

$$\delta\mathcal{J}(u^0) = \int_{\mathbb{R}\setminus[x^-,x^+]} p(x,0)\delta u^0(x)dx$$

This can be done, for instance, if we assume that $p(x,0)$ is constant in $[x^-, x^+]$, $\delta\varphi^0 \equiv 0$ and $\int_{x^-}^{x^+} \delta u^0(x)dx = 0$.

Conclusions

- Really good performance, as for the ADM.
- The creation of extra oscillations due to $\delta\varphi^0$ needs a better control.
  - Consider only linear perturbations: $\delta\varphi^0(x) = \alpha x + \beta$.
  - Use compactly supported functions to restrict the perturbation.
- Modify δu^0 in $[x^-, x^+]$ so that it adapts better to the quasi-shock.
  - Consider perturbations of the form: $\delta u^0(x) = Mu^0(x) + N$.

Thanks for your attention!