The Alternating Descent Method applied to the Augmented Burgers Equation

Task D: Numerical methods for flow control in the presence of shocks
Subtask D.e: Possible use of these techniques for viscous systems

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Outline

Statement of the problem

We want to solve this optimization problem:

$$\mathcal{J}(u^0) = \min_{u^0 \in \mathcal{U}_{ad}} \frac{1}{2} \int_{\mathbb{R}} [u(x,T) - u^*(x)]^2 \, dx$$

subject to

$$\begin{cases} u_t = uu_x + \nu u_{xx} + \frac{c}{\theta} \int_{-\infty}^{x} e^{(y-x)/\theta} u_{xx}(y,t) dy, & (x,t) \in \mathbb{R} \times (0,T) \\ u(x, t=0) = u^0(x), & x \in \mathbb{R} \end{cases}$$

Integrating by parts and rearranging the equation:

$$\begin{cases} u_t + [f_{c,\theta}(u)]_x = \nu u_{xx} + \frac{c}{\theta^3} \left(K_\theta * u - \theta u \right), & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = u(x), & x \in \mathbb{R} \end{cases}$$

where $f_{c,\theta}(u) = -\frac{u^2}{2} - \frac{c}{\theta} u$ and $K_\theta(z) := \chi_{z>0}(z) e^{-z/\theta}$.

Linearized equation

Let us consider $u^{\varepsilon,0}(x) := u^0(x) + \varepsilon\delta u^0(x)$. For $\varepsilon > 0$ sufficiently small, u^ε can be written as $u^\varepsilon = u + \varepsilon\delta u + \mathcal{O}(\varepsilon^2)$, where:

$$\begin{cases} \partial_t \delta u + \partial_x \left[f'_{c,\theta}(u)\delta u \right] = \nu \partial_{xx} \delta u + \frac{c}{\theta^3}\left(K_\theta * \delta u - \theta \delta u \right), & (x,t) \in \mathbb{R} \times (0,T) \\ \delta u(x,0) = \delta u^0(x), & x \in \mathbb{R} \end{cases}$$

In this case, the Gateaux derivative of $\mathcal{J}$ is:

$$\delta \mathcal{J}(u^0)[\delta u^0] = \lim_{\varepsilon \to 0} \frac{\mathcal{J}(u^{\varepsilon,0}) - \mathcal{J}(u^0)}{\varepsilon} = \int_{\mathbb{R}} \left[u(x,T) - u^*(x) \right] \delta u(x,T) dx$$

Adjoint equation

Multiply by a function $p \in \mathcal{C}_0^1(\mathbb{R} \times [0, T])$ and integrate it on $\mathbb{R} \times [0, T]$. Integrating by parts and rearranging terms:

$$\int_{\mathbb{R}} \delta u(x, T) p(x, T) dx - \int_{\mathbb{R}} \delta u(x, 0) p(x, 0) dx$$
$$+ \int_0^T \int_{\mathbb{R}} \delta u \left[-p_t - f'_{c,\theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3} \left(\bar{K}_\theta * p - p \right) \right] dxdt = 0$$

where $\bar{K}_\theta(z) := K_\theta(-z)$. This leads to the adjoint equation:

$$\begin{cases} -p_t - f'_{c,\theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3} \left(\bar{K}_\theta * p - \theta p \right) = 0, & (x, t) \in \mathbb{R} \times (0, T) \\ p(x, T) = u(x, T) - u^*(x), & x \in \mathbb{R} \end{cases}$$

and, therefore,

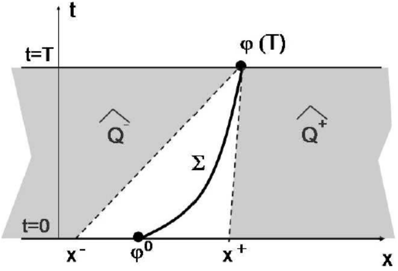
$$\delta \mathcal{J}(u^0)[\delta u^0] = \int_{\mathbb{R}} p(x, T) \delta u(x, T) dx = \int_{\mathbb{R}} p(x, 0) \delta u^0(x) dx$$

which gives the descent direction $\delta u^0(x) = -p(x, 0)$.

Alternating Descent Method

We consider $(u\varepsilon, 0(x), \varphi^{\varepsilon,0}), = (u^0(x), \varphi^0) + \varepsilon(\delta u^0(x), \delta\varphi^0)$ and

$$\begin{cases} u_t + \partial_x [f(u)] = 0, & (x,t) \in Q^- \cup Q^+ \\ \varphi'(t)[u]_{\varphi(t)} = [f(u)]_{\varphi(t)}, & t \in (0,T) \\ \varphi(0) = \varphi^0 & \\ u(x,0) = u^0(x), & x \in \{x < \varphi^0\} \cup \{x > \varphi^0\} \end{cases}$$



Adjoint for the ADM

The Gateux derivative of J is:

$$\delta J = \int_{\{x<\delta\varphi^0\}\cup\{x>\delta\varphi^0\}} p(x,0)\delta u^0(x)dx + q(0)[u^0]_{\varphi^0}\delta\varphi^0$$

where the adjoint state pair (p, q) satisfies the system:

$$\begin{cases} -p_t - f'_{c,\theta}(u)p_x = 0, & (x,t) \in Q^- \cup Q^+ \\ [p]_\Sigma = 0 & \\ q(t) = p(\varphi(t), t), & t \in (0, T) \\ q'(t) = 0, & t \in (0, T) \\ p(x, T) = u(x, T) - u^*(x), & x \in \{x < \varphi(T)\} \cup \{x > \varphi(T)\} \\ q(T) = \frac{\frac{1}{2}\left[(u(x,T)-u^*(T))^2\right]_{\varphi(T)}}{[u]_{\varphi(T)}} & \end{cases}$$

This gives the descent direction $(\delta u^0, \delta\varphi^0) = (-p(x,0), -q(0)[u]_{\varphi^0})$

Augmented Burgers Equation

The Augmented Burgers equation

$$\begin{cases} u_t + [f_{c,\theta}(u)]_x = \nu u_{xx} + \frac{c}{\theta^3} \int_{-\infty}^{x} e^{(y-x)/\theta} [u(y,t) - u(x,t)]\, dy, & (x,t) \in \mathbb{R} \times (0,T) \\ u(x,0) = u(x), & x \in \mathbb{R} \end{cases}$$

is discretized to:

$$\begin{cases} u_k^{n+1} = u_k^n - \dfrac{\Delta t}{\Delta x} \left[g^{EO}(u_k^n, u_{k+1}^n) - g^{EO}(u_{k-1}^n, u_k^n) \right] \quad \text{(flux)} & \\ \qquad + \nu \dfrac{\Delta t}{\Delta x^2} \left(u_{k-1}^n - 2u_k^n + u_{k+1}^n \right) \quad \text{(laplacian)} & \\ \qquad + \dfrac{c}{\theta^3} \Delta t \Delta x \displaystyle\sum_{j=k-N}^{k} e^{\Delta x (j-k)/\theta} \left(u_j^n - u_k^n \right) \quad \text{(integral)} & k \in \mathbb{Z}, t > 0 \\ u_k^0 = \frac{1}{\Delta x} \int_{x_{k-1/2}}^{x_{k+1/2}} u^0(x) dx, & k \in \mathbb{Z} \end{cases}$$

where:

$$g^{EO}(u,v) = \frac{1}{2} \left[f(u) + f(v) - \int_u^v |f'(\xi)| d\xi \right]$$

Adjoint equation and functional

From the numerical scheme of the ABE, the linearized scheme is deduced and from this we conclude:

$$\begin{cases} p_k^n = p_k^{n+1} + \dfrac{\Delta t}{\Delta x}\left[\partial_1 g^{EO}(u_k^n, u_{k+1}^n)(p_{k+1}^{n+1} - p_k^{n+1}) \right. \\ \qquad\qquad \left. + \partial_2 g^{EO}(u_{k-1}^n, u_k^n)(p_k^{n+1} - p_{k-1}^{n+1})\right] \\ \qquad + \nu \dfrac{\Delta t}{\Delta x^2}\left(p_{k-1}^{n+1} - 2p_k^{n+1} + p_{k+1}^{n+1}\right) \\ \qquad + \dfrac{c}{\theta^3}\Delta t \Delta x \displaystyle\sum_{j=k}^{k+N} e^{\Delta x(k-j)/\theta}\left(p_j^{n+1} - p_k^{n+1}\right) \quad k \in \mathbb{Z}, t > 0 \\ p_k^M = u_k^M - u_k^*, \qquad k \in \mathbb{Z} \end{cases}$$

which is, precisely, the upwind scheme for

$$\begin{cases} -p_t - f'_{c,\theta}(u)p_x - \nu p_{xx} - \frac{c}{\theta^3}\left(\bar{K}_\theta * p - p\right) = 0, & (x,t) \in \mathbb{R} \times (0,T) \\ p(x,T) = u(x,T) - u^*(x), & x \in \mathbb{R} \end{cases}$$

This leads to:

$$\mathcal{J}_\Delta(u_{0,\Delta}) = \frac{\Delta x}{2}\sum_{k\in\mathbb{Z}}\left(u_k^M - u_k^*\right)^2 \Rightarrow \delta\mathcal{J}_\Delta = \Delta x \sum_{k\in\mathbb{Z}}\left(u_k^M - u_k^*\right)\delta u_k^M = \Delta x \sum_{k\in\mathbb{Z}} \delta u_k^0 p_k^0$$

Example

Matlab simulation

Conclusions

- ADM obtains a better performance in the presence of shocks.
- Better understanding of influence of the integral kernel.
- For the viscous case, there are no shocks. Consider perturbations of the form $u^{\varepsilon,0}(x) = u^0(x + \varepsilon\delta\varphi(x)) + \varepsilon\delta u^0(x)$.

Discussion

Perturbed Augmented Burgers equation:

$$\begin{cases} u_t^\varepsilon + [f_{c,\theta}(u^\varepsilon)]_x = \nu u_{xx}^\varepsilon + \frac{c}{\theta^3}\left(K_\theta * u^\varepsilon - \theta u^\varepsilon\right), & (x,t) \in \mathbb{R} \times (0,T) \\ u^\varepsilon(x,0) = u^{\varepsilon,0}(x) \end{cases}$$

Linearized system:

$$\begin{cases} (\delta\varphi u_x)_t + \left[(\delta\varphi u_x) f'_{c,\theta}(u)\right]_x - \nu(\delta\varphi u_x)_{xx} - \frac{c}{\theta^3}\left[K_\theta * (\delta\varphi u_x) - \theta(\delta\varphi u_x)\right] = 0 \\ \delta\varphi(x,0) = \delta\varphi^0(x) u_x^0(x) \end{cases}$$

Adjoint system:

$$\begin{cases} -p_t - f'_{c,\theta}(u) p_x - \nu p_{xx} - \frac{c}{\theta^3}\left(\bar{K}_\theta * p - p\right) = 0 \\ p(x,T) = u(x,T) - u^*(x) \end{cases}$$

Gateaux derivative of J:

$$\delta\mathcal{J}(u^0) = \int_{\mathbb{R}} \delta\varphi^0(x) u_x^0(x) p(x,0)\, dx$$

Thanks for your attention!