A rigorous computational method to enclose the eigendecomposition of interval matrices

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Why Interval entries?

- **Uncertainty** of measurement - inaccuracy of data collection
- To consider **variability** of natural phenomena
- Qualitative indicators with **statistical** ranges
- **Errors** in computations
- ....
Computation of bundles of periodic orbits

\[
\begin{align*}
\dot{y} &= g(y) \\
\gamma(t + T) &= \gamma(t) \\
\Gamma &= \{\gamma(t) : t \in [0, T]\} \\
\gamma_\theta(t) &= \gamma(t + \theta)
\end{align*}
\]
Computation of bundles of periodic orbits

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\[ \Gamma = \{ \gamma(t) : t \in [0, T) \} \]
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Linearize around the periodic orbit: \( y' = A_\theta(t)y \)

\[ A_\theta(t) = \nabla g(\gamma_\theta(t)) : T - \text{periodic matrix function} \]

Consider \( \Phi_\theta(t) \) a fundamental matrix solution of the non-autonomous linear system

\[ \begin{cases} 
Y' = A_\theta(t)Y \\
Y(0) = I 
\end{cases} \]

Definition: The monodromy matrix is defined to be \( \Phi_\theta(T) \).

Definition: The eigenvalues of \( \Phi_\theta(T) \) are called the Floquet multipliers.

\[ \dim W^s(\Gamma) = \text{number of Floquet multipliers with modulus less than one.} \]
\[ \dim W^u(\Gamma) = \text{number of Floquet multipliers with modulus greater than one.} \]
Computation of bundles of periodic orbits
The goal of rigorous computations is to construct algorithms that provide an approximate solution to the problem $f(x)=0$ together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

**Ingredients**
Rigorous Computations

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1. Smoothness of the solutions
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2. Spectral methods, choice of suitable Banach spaces
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6. Interval Arithmetic
Some Techniques

Banach Fixed Point Argument

Topological Arguments

Rigorous integration
Let \((B, \| \cdot \|_B)\) be a Banach space, that is a complete normed vector space. Assume the existence of a contraction mapping \(T\) on \(B\), that is a mapping such that

1. \(T(B) \subset B\);

2. there exists \(0 < \kappa < 1\) such that, for every \(x, y \in B\),

\[
\|T(x) - T(y)\|_B \leq \kappa \|x - y\|_B.
\]

Then there exists a unique \(x^* \in B\) such that \(T(x^*) = x^*\).
Rigorous Computations: **Banach Fixed Point Theorem**

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---

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can be verified using interval arithmetic!
Rigorous computations: **Formulation in Banach Space**
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\[ F(u, \nu) = 0 \]

(E.g: Differential Equations)

Knowledge about regularity

\[ x \in \Omega^s = \left\{ (x_k)_k : \sup_k \omega^s_k \|x_k\| < \infty \right\} \]

Spectral methods

\[ f(x, \nu) = 0 \]

\[ x : \text{modes} \]

\[ \nu : \text{parameter} \]
Rigorous computations: **Formulation in Banach Space**

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Consider \( \bar{x} \) such that \( f^m(\bar{x}, \nu_0) \approx 0 \).

Finite dim. reduction

\[ f(x, \nu) = 0 \iff T_\nu(x) = x \]

Newton-like operator at \( \bar{x} \)
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**Finite dim. reduction**

\[ f(x, \nu) = 0 \iff T_\nu(x) = x \]

**Newton-like operator** at \( \bar{x} \)

\[ T_\nu : \Omega^s \to \Omega^s \]

\[ T_\nu(x) = x - J f(x, \nu) \]

\[ J \approx D_x f(\bar{x}, \nu_0)^{-1} \]

The chances of contracting a small set \( B \) around \( \bar{x} \) depends on the magnitude of the eigenvalues of \( J \).
Q: How to find a ball $B_{\bar{x}}(r)$ such that $T_v: B_{\bar{x}}(r) \rightarrow B_{\bar{x}}(r)$ is a contraction?
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$B_{\bar{x}}(r) = \bar{x} + B(r)$
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$B_{\bar{x}}(r) = \bar{x} + B(r)$ Ball of radius $r$
centered at 0
in the space $\Omega^s$

A: Radii polynomials

Suppose the bounds $Y, Z(r)$ satisfy

$$
\left| \left[ T_\nu(\bar{x}) - \bar{x} \right]_k \right| \leq Y \quad \quad \sup_{b,c \in B(r)} \left| \left[ D_x T_\nu(\bar{x} + b) c \right]_k \right| \leq Z(r)
$$

$$
p_k(r) = Y + Z(r) - \frac{r}{w^s_k}
$$
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Ball of radius $r$ centered at $0$ in the space $\Omega^s$

A: **Radii polynomials**

Suppose the bounds $Y, Z(r)$ satisfy

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$$p_k(r) = Y + Z(r) - \frac{r}{w^s_k}$$

**Lemma:** If there exists $r > 0$ such that $p_k(r) < 0$ for all $k$, then there is a unique $\hat{x} \in B_{\bar{x}}(r)$ s.t. $f(\hat{x}, \nu) = 0$.

**proof.** Banach fixed point theorem.
Rigorous Computations

An Example: Compute solution of \( x^2 - 2 = 0 \)
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$\Omega = \mathbb{R}$

$\bar{x} = 1.41$

✓ Choose the Banach Space

✓ Compute num. solution $\bar{x}$

Define the fixed point operator

Define $\sup_{b,c \in B(r)} \left| \left[ D_x T(\bar{x} + b)c \right]_k \right| \leq Z(r)$

Define radii polynomial

Find $r$
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\[ \sup_{b, c \in B(r)} \left| \frac{T(\bar{x}) - \bar{x}}{k} \right| \leq Y \]

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Find $r$
An Example: Compute solution of \( x^2 - 2 = 0 \)

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\begin{align*}
\Omega &= \mathbb{R} \\
\bar{x} &= 1.41 \\
Df(\bar{x})^{-1} &\approx 0.35 \\
T(x) &= x - 0.35(x^2 - 2)
\end{align*}
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\[ |T(\bar{x}) - \bar{x}| \leq 0.0042 \]

\[ |DT(\bar{x} + b)c| \leq 0.013 r + 0.7 r^2 \]

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\[ p(r) = 0.7r^2 - 0.987r + 0.0042 \]

✓ Choose the Banach Space
✓ Compute num. solution \( \bar{x} \)
✓ Define the fixed point operator
✓ Define \( \sup_{b,c \in B(r)} |[T(x) - \bar{x}]_k| \leq Y \)
✓ Define radii polynomial
Find \( r \)

\[ [D_x T(\bar{x} + b)c]_k \leq Z(r) \]
Rigorous Computations

An Example: Compute solution of \( x^2 - 2 = 0 \)

\[ \Omega = \mathbb{R} \]
\[ \bar{x} = 1.41 \]
\[ \begin{aligned} \frac{Df}{Df}(\bar{x})^{-1} & \approx 0.35 \\ T(x) &= x - 0.35(x^2 - 2) \\ \left| T(\bar{x}) - \bar{x} \right| & \leq 0.0042 \\ \left| DT(\bar{x} + b)c \right| & \leq 0.013r + 0.7r^2 \\ p(r) &= 0.7r^2 - 0.987r + 0.0042 \end{aligned} \]

✓ Choose the Banach Space
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$|DT(\bar{x} + b)c| \leq 0.013r + 0.7r^2$

$p(r) = 0.7r^2 - 0.987r + 0.0042$

$I = [0.00427, 1.4057], \forall r \in I, p(r) < 0$ 

$\sqrt{2} \in 1.41 \pm 0.00427$
Eigendecomposition of interval matrices

Notation: **bold case** $\longrightarrow$ interval

normal case $\longrightarrow$ scalar

$A \in \mathbb{A}$

$A_{i,j} = [a_{i,j} \ \bar{a}_{i,j}]$, $\hat{A}$ is the *center* of $A$
Eigendecomposition of interval matrices

Notation: **bold case** $\rightarrow$ interval
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**GOAL: enclosing eigendecomposition**

Given $A \in \mathbb{I\mathbb{C}}^{n \times n}$

construct a set of balls $B_i \in \mathbb{C} \times \mathbb{C}^n, \ i = 1 \ldots n$

such that

for any $A \in \mathbb{A}$ and any $i = 1 \ldots n$

there exists $x_i = (\lambda_i, v_i) \in B_i$ s.t $A v_i = \lambda_i v_i$,

where $x_i$ is unique up to a scaling factor of $v_i$. 
Some literature


Rigorous computation of eigendecomposition: scalar case

Define the problem $f(x) = 0$
Rigorous computation of eigendecomposition: scalar case

Define the problem $f(x) = 0$

Suppose $(\bar{\lambda}, \bar{v})$ is an approximate eigenpair of $A$, i.e. $A\bar{v} \approx \bar{\lambda}\bar{v}$

$$f(x) : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$x = (\lambda, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n)$$

$$f(x) = 0 \iff (\lambda, v) \text{ is an eigenpair of } A.$$
Rigorous computation of eigendecomposition: scalar case

Define the problem $f(x)=0$

Suppose $(\bar{\lambda}, \bar{v})$ is an approximate eigenpair of $A$, i.e. $A\bar{v} \approx \bar{\lambda}\bar{v}$

$$f(x) : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$x = (\lambda, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n)$$

$$\Downarrow$$

$$f(x) = A \begin{bmatrix} v_1 \\ \vdots \\ \bar{v}_k \\ \vdots \\ v_n \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ \vdots \\ \bar{v}_k \\ \vdots \end{bmatrix}, \quad \bar{v}_k = \max\{|\bar{v}_i|\}$$

$$f(x) = 0 \Leftrightarrow (\lambda, v) \text{ is an eigenpair of } A.$$
Rigorous computation of eigendecomposition: scalar case

Numerical solution: \( \bar{x} = (\bar{\lambda}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}, \bar{v}_{k+1}, \ldots, \bar{v}_n) \)

Banach space: \( \Omega = \{ x \in \mathbb{C}^n : ||x||_\Omega < \infty \}, \quad ||x||_\Omega = \max_i |x_i|, \)

Operator \( T(x) \): \( T(x) = x - Rf(x) \)

\[
R \approx Df(\bar{x})^{-1}, \quad Df(\bar{x}) = \begin{bmatrix}
\bar{v}_1 \\
\vdots \\
\bar{v}_k \\
\vdots \\
\bar{v}_n
\end{bmatrix} (A - I_n) \hat{k}
\]
Rigorous computation of eigendecomposition: scalar case

Definition of the bounds $Y, Z(r)$

$(T(x) = x - Rf(x))$
Rigorous computation of eigendecomposition: scalar case

Definition of the bounds $Y, Z(r)$

\[ Y \geq |T(\bar{x}) - \bar{x}| \rightarrow Y := |Rf(\bar{x})| \]
Rigorous computation of eigendecomposition: scalar case

Definition of the bounds $Y$, $Z(r)$

$(T(x) = x - Rf(x))$

$Y \geq |T(\bar{x}) - \bar{x}| \rightarrow Y := |Rf(\bar{x})|$

$Z(r) \geq \sup_{b, c \in B(r)} |DT(\bar{x} + b)c|$

$|DT(\bar{x} + b)c| \leq |(I_n - R \cdot Df(\bar{x}))c| + |R[(Df(\bar{x}) - Df(\bar{x} + b))c]|$

$\leq r|I_n - R \cdot Df(\bar{x})|1_n + r^2 2|R|\hat{1}_n \quad (|b|, |c| \leq r)$

$Z(r) := rZ_0 + r^2Z_1$
Rigorous computation of eigendecomposition: scalar case

Definition of the bounds $Y, Z(r)$

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Z(r) \geq \sup_{b,c \in B(r)} |DT(\bar{x} + b)c| \\
|DT(\bar{x} + b)c| \leq |(I_n - R \cdot Df(\bar{x}))c| + |R[(Df(\bar{x}) - Df(\bar{x} + b))c]| \\
\leq r|I_n - R \cdot Df(\bar{x})|1_n + r^22|R|\hat{1}_n \quad (|b|, |c| \leq r) \\
Z(r) := rZ_0 + r^2Z_1 \\
p_k(r) = (Y + Z(r))_k - r.
\]
Rigorous computation of eigendecomposition: scalar case

Definition of the bounds \( Y, Z(r) \)

\[
Y \geq |T(\bar{x}) - \bar{x}| \rightarrow Y := |Rf(\bar{x})|
\]

\[
Z(r) \geq \sup_{b, c \in B(r)} |DT(\bar{x} + b)c|
\]

\[
|DT(\bar{x} + b)c| \leq |(I_n - R \cdot Df(\bar{x}))c| + |R[(Df(\bar{x}) - Df(\bar{x} + b))c]| \leq r|I_n - R \cdot Df(\bar{x})|1_n + r^2 2|R|\hat{1}_n \quad (|b|, |c| \leq r)
\]

\[
Z(r) := rZ_0 + r^2Z_1
\]

\[
p_k(r) = (Y + Z(r))_k - r.
\]

Suppose that for any \( r \in I, p_k(r) < 0 \) for any \( k \).
Then, for any \( r \in I, \) there exists a genuine eigenpair \( x = (\lambda, v) \) of \( A \) in the ball \( B_{\bar{x}}(r) \) around \((\bar{\lambda}, \bar{v})\)
Rigorous computation of eigendecomposition: extension to interval matrix

Consider interval matrix $\mathbf{A}$ and interval arithmetics
Consider interval matrix $A$ and interval arithmetics

$$A \in \mathbb{IC}^{n \times n}, \quad f : \mathbb{C}^n \rightarrow \mathbb{IC}^n, \quad Df \in \mathbb{IC}^{n \times n}$$
Rigorous computation of eigendecomposition: extension to interval matrix

Consider interval matrix $A$ and interval arithmetics

$$A \in \mathbb{I}\mathbb{C}^{n\times n}, f : \mathbb{C}^n \rightarrow \mathbb{I}\mathbb{C}^n, Df \in \mathbb{I}\mathbb{C}^{n\times n}$$

Define $\bar{x} = (\bar{\lambda}, \bar{v})$ a numerical eigenpair of $\hat{A}$

Let $R$ be a numerical inverse of $\hat{Df}$ and define $T = x - Rf(x)$
Rigorous computation of eigendecomposition: extension to interval matrix

Consider interval matrix $\mathbf{A}$ and interval arithmetics $\mathbf{A} \in \mathbb{IC}^{n \times n}$, $\mathbf{f} : \mathbb{C}^n \rightarrow \mathbb{IC}^n$, $\mathbf{Df} \in \mathbb{IC}^{n \times n}$

Define $\bar{x} = (\bar{\lambda}, \bar{v})$ a numerical eigenpair of $\hat{\mathbf{A}}$

Let $R$ be a numerical inverse of $\hat{\mathbf{Df}}$ and define $\mathbf{T} = x - R\mathbf{f}(x)$

$p_k(r) < 0$

\[\Downarrow\]

for any $A \in A$ there exists a unique $(\lambda, v)$ so that

$|\lambda - \bar{\lambda}| < r$, $|v_j - \bar{v}_j| < r$, $v_k = \bar{v}_k$ and $Av = \lambda v$. 
The Matlab Code

1. \textbf{function} \texttt{r=enc_eig(A,V,l)}
2. \texttt{n= size(V,1);}
3. \texttt{iV=intval(V);}
4. \texttt{[m,IV]=max(abs(V));}
5. \texttt{O=ones(n,1);}
6. \texttt{df=intval(A)-intval(l)*speye(n);}
7. \texttt{F=df*iV;}
8. \texttt{df(:,IV)=-iV;}
9. \texttt{R=inv(mid(df));}
10. \texttt{Y=sup(abs(R*F));}
11. \texttt{Z0=sup(abs(eye(n)-R*df))*O;}
12. \texttt{O(IV)=0;}
13. \texttt{Z1=2*abs(R)*O;}
14. \texttt{rd=[Z1,Z0-ones(n,1),Y];} \quad % \text{coefficients of radii polynomials \{r^2,r,const\}}
15. \texttt{r=evaluate_intersection_int(rd); \quad % \text{evaluate where all the polynomials are negative}}
16. \texttt{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%}
17. \textbf{function} \texttt{neg_inter=evaluate_intersection_int(rd)}
18. \texttt{i2=intval(2);}
19. \texttt{rd=intval(rd);}
20. \texttt{neg_inter=NaN;}
21. \texttt{Rm=(-rd(:,2)-sqrt(rd(:,2).^i2-intval(4)*rd(:,1).*rd(:,3)))./(i2*rd(:,1));}
22. \texttt{Rp=-Rm-rd(:,2)./rd(:,1);}
23. \texttt{ISCOMP=find(isreal(Rp)+isreal(Rm)<2, 1);}
24. \texttt{if isempty(ISCOMP)==0}
25. \quad \text{'error: a polynomial is positive everywhere'}
26. \quad \texttt{return}
27. \texttt{end}
28. \texttt{intersection=[max(Rm.sup),min(Rp.inf)];}
29. \texttt{if intersection(1)<intersection(2)}
30. \quad \texttt{neg_inter=intersection(1);}
31. \texttt{end}
32. \texttt{end}
33. \texttt{end}
34. \texttt{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%}
35. 

Definition of radii polynomials

Compute intersections of intervals
Some computational results

Enclosure of one eigenpair for large matrices

\[ D = \text{diag}(-N/2, \ldots, N/2), \quad X = \text{rand(size}(D)), \quad A = X \cdot D \cdot X^{-1} \]

\[ [q, w] = \text{eig}(A) \rightarrow \bar{x} = (q(:, k), w(k, k)) \]

<table>
<thead>
<tr>
<th>N</th>
<th>r</th>
<th>time (s)</th>
</tr>
</thead>
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<tr>
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Some computational results

Enclosure of the Eigendecomposition for large intervals

\[
D = \text{diag}(0, e^{ik \frac{2\pi}{5}}), \quad X = \text{rand(size}(D)) + i \text{rand(size}(D)) \\
\hat{A} = X \cdot D \cdot X^{-1}, \quad A = \hat{A} + 1 rad
\]
Some computational results

Comparison of the computational time

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<tr>
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<th>N=10</th>
<th>N=50</th>
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Computational time (s)

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<th>verifyeig</th>
<th>verifyynlss</th>
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</thead>
<tbody>
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<tr>
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<td>0.0650</td>
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<tr>
<td>500</td>
<td>377.3</td>
<td>423.6</td>
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</table>

verifyeig - verification method, Brouwer’s fix p.
verifyynlss - Krawczyk operator for the computation of zeros of nonlinear functions

$\hat{A}$ is a matrix with eigenvalue 0 and $N$ equispaced on the unit circle.

On the left $rad = 10^{-15}$
Some computational results

Comparison of the computational time

Our approach is satisfactory from the point of view of accuracy of the result and the algorithm is faster then verifyeig, especially for small $N$. This achievement has been possible thanks to the analytical estimates, in the form of radii polynomials, that allow minimizing the number of computations done with intervals.
Thank you
Köszönöm   -   Eskerrik asko
for your attention