NUMERIWAVES Meeting

March 14, 2014

Outline

Kolmogorov equation on a bounded domain

The equation

$$\partial_t f = \partial_v^2 f - v\partial_x f \qquad (t,v,x) \in \mathbb{R}_+^* \times \Omega_v \times \Omega_x \,, \tag{1a}$$

$$f(0,v,x) = f_0(v,x) \qquad (v,x) \in \Omega_v \times \Omega_x \,, \tag{1b}$$

where Ω_v is a bounded interval of $\mathbb{R}$ and $f = 0$ on $\partial\Omega$, $\Omega_x = \mathbb{T} = \mathbb{R}/(2\pi)$ and the initial condition f_0 is given in $L^2(\Omega_v \times \Omega_x)$.

$$f(t,v,0) = f(t,v,2\pi) \qquad ((t,v) \in \mathbb{R}_+ \times \Omega_v) \,, \tag{2}$$

Kolmogorov equation on a bounded domain

Decay Theorem

There exists a unique solution f in $C(\mathbb{R}_+, L^2(\Omega_v \times \mathbb{T}))$. In addition, there exists $\kappa > 0$ (depending only of Ω_v) such that:

$$\|f(t)\|_{L^2(\Omega_v \times \mathbb{T})} \leqslant \exp(-\kappa t)\|f_0\|_{L^2(\Omega_v \times \mathbb{T})} \qquad (t \in \mathbb{R}_+)\,. \tag{3}$$

Numerics for Kolmogorov equation on a bounded domain

Algorithm

Let $\tau > 0$ be the time step and $f^n(v, x)$ be the approximation of $f(n\tau, v, x)$. For every $n \in \mathbb{N}$, we define the next time evaluation f^{n+1} by:

$$\frac{1}{\tau}\left(f^{n+\frac{1}{2}} - f^n\right) = \theta\partial_v^2 f^n + (1-\theta)\partial_v^2 f^{n+\frac{1}{2}}\,, \tag{4a}$$

$$f^{n+1}(v, x) = f^{n+\frac{1}{2}}(v, x - \tau v) \qquad (v, x) \in \Omega_v \times \Omega_x\,, \tag{4b}$$

where $\theta \in [0, 1]$ is fixed.

Numerics for Kolmogorov equation on a bounded domain

Exponential Decay Rate

Let $\theta \in [0,1]$ and $(f^n)_{n\in\mathbb{N}}$ be the solution of (4) with f^0 given in $L^2(\Omega_v \times \Omega_x)$. Then there exists $\kappa > 0$ such that:

$$\|f^n\|_{L^2(\Omega_v\times\Omega_x)} \leqslant \exp(-\kappa n\tau)\|f^0\|_{L^2(\Omega_v\times\Omega_x)} \qquad (n \in \mathbb{N})\,.$$

Numerics for Kolmogorov equation on a bounded domain

Numerical Results

$\Omega_v = [-2\pi, 2\pi]$, $\Omega_x = \mathbb{T}$, and $t \in I = [0, 2]$.
In the first test we use the initial condition

$$f_0(v,x) = A\, e^{-\frac{1}{2\sigma}\left((v-v_0)^2+(x-x_0)^2\right)}, \tag{5}$$

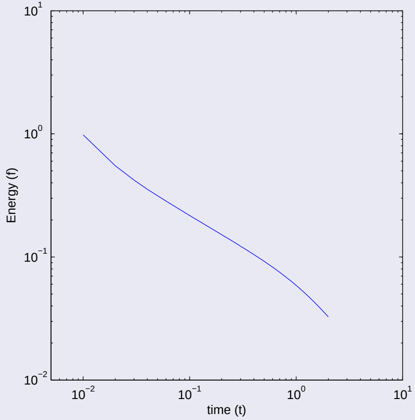
$A = 10$, $\sigma = 1 \times 10^{-4}$, $v_0 = -\pi$, and $x_0 = \pi$.In the second test

$$f_0(v,x) = A\, e^{-\frac{1}{2\sigma}\left((v-v_0)^2+(x-x_0)^2\right)} + A\, e^{-\frac{1}{2\sigma}\left((v+v_0)^2+(x-x_0)^2\right)}. \tag{6}$$

The numerical discretization was achieved by the Finite Element method (FEM), where the element was the standard P_1 Lagrange Finite Element (FE). The triangulation consisted a uniform triangulation with 128 triangles in the x-directions and 128 triangles in the v-direction, for a total of 16 384 triangles. $\Delta t = 0.01$.

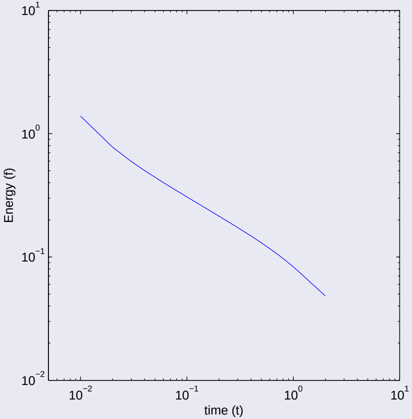
Numerics for Kolmogorov equation on a bounded domain

Decay Rate for the Single Gaussion Initial Condiiton



Numerics for Kolmogorov equation on a bounded domain

Decay Rate for the Double Gaussion Initial Condiiton



Kolmogorov equation on the full space

The equation

$$\partial_t u - \partial_v^2 u - v\partial_x u = 0 \qquad (t,x,y) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}\,, \tag{7a}$$

$$u(0,x,y) = u_0(x,y) \qquad (x,y) \in \mathbb{R}^2\,, \tag{7b}$$

Kolmogorov equation on the full space

Theorem

For every $f_0 \in C_c^\infty(\mathbb{R}^2)$, the solution f is given by:

$$f(t,v,x) = \left(f_0 * \mathcal{F}^{-1}\left(\exp\left(\frac{-t}{3}(3\nu^2 + 3\nu t\zeta + t^2\zeta^2)\right)\right)\right)(v, x+tv) \quad (8)$$

Which is equivalent to:

$$f(t,v,x) = (f_0 * G_t)(v, x+tv) \quad (9)$$

with:

$$G_t(v,z) = \frac{\sqrt{3}}{2\pi t^2}\exp\left(\frac{-1}{4t^3}\left(3z^2 + (2tv - 3z)^2\right)\right).$$

Kolmogorov equation on the full space

Sketch of the proof

$$f(t, v, x) = g(t, v, x + tv) = g(t, v, z)\,.$$

Kolmogorov equation on the full space

Sketch of the proof

$$f(t,v,x) = g(t,v,x+tv) = g(t,v,z)\,.$$

Sketch of the proof

Consequently, g is solution of

$$\partial_t g = \partial_v^2 g + 2t\partial_v\partial_z g + t^2\partial_z^2 g \quad (t,v,z) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}\,,$$
$$g(0,v,z) = f_0(v,z) \quad (v,z) \in \mathbb{R}^2\,,$$

Kolmogorov equation on the full space

Sketch of the proof

Define $\hat{g}(t,\nu,\zeta) = \int_{\mathbb{R}^2} g(t,v,z)e^{-i(\nu z+\zeta z)}\,dvdz$. Then $\hat{g}$ is solution of:

$$\partial_t \hat{g} = -\left(\nu^2 + 2t\nu\zeta + t^2\zeta^2\right)\hat{g}(t,\nu,\zeta) \quad (\nu,\zeta)\in\mathbb{R}_+^*\times\mathbb{R}^2\,,$$
$$\hat{g}(0,\nu,\zeta) = \widehat{f_0}(\nu,\zeta) \quad (\nu,\zeta)\in\mathbb{R}^2\,.$$

Kolmogorov equation on the full space

Sketch of the proof

Define $\hat{g}(t,\nu,\zeta) = \int_{\mathbb{R}^2} g(t,v,z)e^{-i(\nu z+\zeta z)}\,dvdz$. Then $\hat{g}$ is solution of:

$$\begin{aligned} \partial_t \hat{g} &= -\left(\nu^2 + 2t\nu\zeta + t^2\zeta^2\right)\hat{g}(t,\nu,\zeta) \quad && (\nu,\zeta) \in \mathbb{R}_+^* \times \mathbb{R}^2, \\ \hat{g}(0,\nu,\zeta) &= \widehat{f_0}(\nu,\zeta) && (\nu,\zeta) \in \mathbb{R}^2. \end{aligned}$$

Sketch of the proof

$$\hat{g}(t,\nu,\zeta) = \widehat{f_0}(\nu,\zeta)\exp\left(\frac{-t}{3}(3\nu^2 + 3\nu t\zeta + t^2\zeta^2)\right).$$

Kolmogorov equation on the full space

Decay rate

Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f_0 \in L^p(\mathbb{R}^2)$, then the solution f of (7) satisfies for every $t > 0$, $f(t) \in L^r(\mathbb{R}^2)$, and:

$$\|f(t)\|_{L^r(\mathbb{R}^2)} \leqslant \begin{cases} \dfrac{C(q)}{t^{2(1-\frac{1}{q})}} \|f_0\|_{L^p(\mathbb{R}^2)} & \text{if } q \in [1, \infty)\,, \\ \dfrac{C(q)}{t^2} \|f_0\|_{L^p(\mathbb{R}^2)} & \text{if } q = \infty\,. \end{cases} \tag{10}$$

The proof uses Young's inequality.

Kolmogorov equation on the full space

Leslie Greengard work on the heat equation

- L. Greengard and J. Strain. A fast algorithm for the evaluation of heat potentials. Comm. Pure Appl. Math., 43(8):949-963, 1990.
- L. Greengard and J. Strain. The fast Gauss transform. SIAM J. Sci. Statist. Comput., 12(1):79-94, 1991.
- L. Greengard and P. Lin. Spectral approximation of the free-space heat kernel. Appl. Comput. Harmon. Anal., 9(1):83-97, 2000.
- J.-R. Li and L. Greengard. High order accurate methods for the evaluation of layer heat potentials. SIAM J. Sci. Comput., 31(5):3847- 3860, 2009.

Kolmogorov equation on the full space

Idea: Approximate the heat kernel

$$G(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-s^2 t} e^{isx}\, ds.$$

Kolmogorov equation on the full space

Idea: Approximate the heat kernel

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Lemma: Greengard (2000)

Let ϵ be the precision we need. For $t \geq \delta > 0$

$$\left| \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} - \frac{1}{2\pi}\int_{-P}^{P} e^{-s^2 t} e^{isx} ds \right| \leq \frac{e^{-P\delta^2}}{\sqrt{4\pi\delta}} = \epsilon.$$

Kolmogorov equation on the full space

Theorem (Greengard and Lin, 2000)

Let ϵ be the precision we need, $[a, b] = [2^j, 2^{j+1}]$. If we choose $n(j) = \max\{R2^{j+1}, 4log(1/\epsilon)\}$, then if we use the classical Gauss-Legendre quarature, then

$$\left| \int_a^b e^{-s^2 t} e^{isx} ds - \sum_{k=1}^{n(j)} e^{-s_k^2 t} e^{is_k x} w_k \right| \leq \frac{(b-a)}{P} \epsilon.$$

Kolmogorov equation on the full space

Theorem (Greengard and Lin, 2000)

Let ϵ be the precision we need

$$\left| \int_{-P}^{P} e^{-s^2 t} e^{isx} ds - \sum_{j=log(\epsilon)}^{log(P)} \sum_{k=1}^{n(j)} e^{-s_k^2 t} e^{is_k x} w_k \right| \leq \epsilon.$$

Kolmogorov equation on the full space

Theorem (Greengard and Lin, 2000)

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Theorem (Greengard and Lin, 2000)

$$\left| \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} - \sum_{j=log(\epsilon)}^{log(P)} \sum_{k=1}^{n(j)} e^{-s_{k,j}^2 t} e^{is_{k,j} x} w_k \right| = O(\epsilon).$$

Kolmogorov equation on the full space

The heat equation

$$\partial_t u - \partial_x^2 u = 0 \qquad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}\,, \tag{11a}$$

$$u(0, x) = u_0(x) \qquad x \in \mathbb{R}\,, \tag{11b}$$

Kolmogorov equation on the full space

The heat equation

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$$u(0,x) = u_0(x) \qquad x \in \mathbb{R}\,, \tag{11b}$$

Theorem (Greengard and Li, 2007, 2009)

$$\left| u(x,t) - \sum_{j=log(\epsilon)}^{log(P)} \sum_{k=1}^{n(j)} e^{-s_{k,j}^2 t} e^{i s_k x} w_k \hat{u}_0(s_{k,j}) \right| = O(\epsilon).$$

Kolmogorov equation on the full space

Theorem

$$\Bigg|\frac{\sqrt{3}}{2\pi t^2}\exp\left(\frac{-1}{4t^3}\left(3z^2+(2tv-3z)^2\right)\right)$$

$$-\sum_{j,l=L_{min}}^{L_{max}}\sum_{k=1}^{n(j)}\sum_{h=1}^{m(l)}(e^{i\xi_{j,k}z}+e^{-i\xi_{j,k}z})(e^{i\zeta_{l,h}v}+e^{-i\zeta_{l,h}v})w_{j,k}w_{l,h}\Bigg|$$

$$=O(\epsilon).$$

Kolmogorov equation on the full space

Theorem

$$\left| u(t,z,v) - \sum_{j,l=L_{min}}^{L_{max}} \sum_{k=1}^{n(j)} \sum_{h=1}^{m(l)} \right.$$

$$\left. (e^{i\xi_{j,k}z} + e^{-i\xi_{j,k}z})(e^{i\zeta_{l,h}v} + e^{-i\zeta_{l,h}v}) w_{j,k} w_{l,h} \hat{u}_0(\xi_{j,k}, \zeta_{l,h}) \right|$$

$$= O(\epsilon).$$