Panorama on the existence of solutions for compressible and incompressible Navier-Stokes

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In this talk, we are concerned with the following model of incompressible viscous fluid with variable density for $N \geq 2$:

- Mass conservation :

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0,$$

- Momentum conservation :

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D u) + \nabla \Pi = \rho f,$$

- Incompressibility :

$$\operatorname{div} u = 0,$$

- Initial data :

$$(\rho, u)_{/t=0} = (\rho_0, u_0).$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density, $Du = \frac{1}{2}(\nabla u + {}^t \nabla u)$ is the strain tensor. We denote by μ the viscosity coefficients of the fluid, which is assumed to satisfy $\mu > 0$. The term $\nabla \Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\operatorname{div} u = 0$.

We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathbb{T}_a^N$ with period a_i, in the i-th direction. We restrict ourselves the case $N \geq 2$.

We aim at solving in the case where the data (ρ_0, u_0, f) have critical regularity for the scaling of the equations. By critical, we mean that we want to solve the system functional spaces with norm in invariant by the changes of scales which leaves invariant the system. That approach has been initiated by H. Fujita and T. Kato . For density-dependent incompressible fluids, one can check that the appropriate transformation are:

$$(\rho_0(x), u_0(x)) \longrightarrow (\rho_0(lx), lu_0(lx)), \quad \forall\, l \in \mathbb{R}.$$
$$(\rho(t,x), u(t,x), \Pi(t,x)) \longrightarrow (\rho(l^2t, lx), lu(l^2t, lx), l^2\Pi(l^2t, lx)).$$

Some results of strong solutions for (NS)

- Fujita-Kato [60s], strong solutions with $u_0 \in \dot{H}^{\frac{N}{2}-1}$, next $u_0 \in L^N$.
- M. Cannone, Y. Meyer, F. Planchon [95], strong solutions with $u_0 \in \dot{B}^{\frac{N}{p}-1}_{p,r}$, for $1 \leq r \leq +\infty$, $1 \leq p < +\infty$.
- H. Koch et D. Tataru [01], strong solutions with $u_0 \in BMO^{-1}$.
- J. Bourgain and N. Pàvlovic [09], ill-posedness in $B^{-1}_{\infty,\infty}$.

Some results for (NS) with dependent-density

- O. Ladyzhenskaya and V. Solonnikov [70s], under the conditions that $u_0 \in W^{2-\frac{2}{q},q}$ $(q > N)$ and $\text{div}u_0 = 0$
  - Global solution for $N = 2$
  - Strong solutions in finite time for $N \geq 2$.
- R. Danchin [2003], strong solutions in finite time with $(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}, u_0) \in H^{\frac{N}{2}+\varepsilon} \times H^{\frac{N}{2}-1+\varepsilon}$ for $\bar{\rho} > 0$
- R. Danchin [2005], strong solutions in finite time when $(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}, u_0) \in (\dot{B}^{\frac{N}{2}}_{2,\infty} \cap L^\infty) \times \dot{B}^{\frac{N}{2}-1}_{2,1}$ + a condition of smallness on the density.
- H. Abidi and M. Paicu [2006], strong solution in finite time with $(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}, u_0) \in \dot{B}^{\frac{N}{p_1}}_{p_1,1} \times \dot{B}^{\frac{N}{p_2}-1}_{p_2,1}$ + a condition of smallness on the density.

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let (φ,χ) be a couple of smooth functions valued in $[0,1]$, such that φ is supported in the shell supported in $\{\xi\in\mathbb{R}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$, χ is supported in the ball $\{\xi\in\mathbb{R}^N/|\xi|\leq\alpha\}$ such that:

$$\forall\xi\in\mathbb{R}^N,\quad \chi(\xi)+\sum_{l\in\mathbb{N}}\varphi(2^{-l}\xi)=1.$$

Denoting $h=\mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u=\varphi(2^{-l}D)u=2^{lN}\int_{\mathbb{R}^N}h(2^l y)u(x-y)dy \quad\text{with}\quad h=\mathcal{F}^{-1}\varphi,\quad\text{if}\quad l\in\mathbb{Z},$$

$$S_l u=\chi(2^{-l}Du).$$

Formally, one can write that: $u=\sum_{k\in\mathbb{Z}}\Delta_k u$ (modulo the polynoms).

Proposition

For any $u\in\mathcal{S}'(\mathbb{R}^N)$ and $v\in\mathcal{S}'(\mathbb{R}^N)$, we have:

$$\Delta_p\Delta_q u=0\quad\text{if}\quad|p-q|\geq 2\qquad\text{and}\qquad\Delta_q(S_{p-1}u\Delta_p v)=0\quad\text{if}\quad|p-q|\geq 5,$$

Definition

We denote by $\mathcal{S}'_h$ the space of temperate distributions u such that: $\lim_{j\to+\infty}S_j u=0$ in $\mathcal{S}'$.

Definition

The Besov space $B^s_{p,r}$ is the set of temperate distributions $u \in \mathcal{S}'_h$ such that $\|u\|_{B^s_{p,r}} < +\infty$ with:

$$\|u\|_{B^s_{p,r}} = \Big(\sum_{l\in\mathbb{Z}} 2^{lsr}\|\Delta_l u\|^r_{L^p}\Big)^{\frac{1}{r}}$$

Definition

Let $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ et $s_1 \in \mathbb{R}$. We define the Chemin-Lerner spaces as follows:

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1}_{p,r})} = \Big(\sum_{l\in\mathbb{Z}} 2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho_T(L^p)}\Big)^{\frac{1}{r}}.$$

Corollary

Let $r \in [1,+\infty]$, $1 \le p \le p_1 \le +\infty$ and s such that:

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} \le 1$,

then we have if $u \in B^s_{p,r}$ et $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$:

$$\|uv\|_{B^s_{p,r}} \le C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

Idea of the proof: For u and v we have the formal decomposition:

$$uv = T_u v + T_v u + R(u,v),$$

with $T_u v = \sum_{p\leq q-2} \Delta_p u \Delta_q v = \sum_p S_{p-1} u \Delta_p v$ and $R(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v$ where $\widetilde{\Delta}_q v = \Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v$.

Proposition

There exists a constant C such that for any couple of real numbers (s,σ) with σ positive and for any (p,r,r_1,r_2) in $[1,+\infty]^4$ with $1/r = 1/r_1 + 1/r_2$, we have

$$\|\dot{T}\|_{\mathcal{L}(L^\infty \times \dot{B}^s_{p,r};\dot{B}^s_{p,r})} \leq C^{|s|+1}$$

if $s < \frac{N}{p}$ *or* $s = \frac{N}{p}$ *and* $r = 1$, *and*

$$\|\dot{T}\|_{\mathcal{L}(\dot{B}^{-\sigma}_{\infty,r_1} \times \dot{B}^s_{p,r_2};\dot{B}^{s-\sigma}_{p,r})} \leq \frac{C^{|s-\sigma|+1}}{\sigma}.$$

if $s-\sigma < N/p$, *or* $s-\sigma = N/p$ *and* $r=1$.

Proposition

Let $s \in \mathbb{R}$ *and* $1 \leq p, r \leq \infty$ *satisfy*

$$s < \frac{N}{p}, \quad \text{or} \quad s = \frac{N}{p} \quad \text{and} \quad r = 1. \tag{1}$$

Let $(u_q)_{q\in\mathbb{Z}}$ *be a sequence of functions such that*

$$\Big(\sum_{q\in\mathbb{Z}} (2^{qs}\, \|u_q\|_{L^p})^r\Big)^{\frac{1}{r}} < \infty.$$

If $\widehat{u}_q \subset \mathcal{C}(0, 2^q R_1, 2^q R_2)$ *for some* $0 < R_1 < R_2$ *then* $u := \sum_{q\in\mathbb{Z}} u_q$ *belongs to* $B^s_{p,r}$ *and there exists a constant* C *such that*

$$\|u\|_{B^s_{p,r}} \leq C^{1+|s|} \Big(\sum_{q\in\mathbb{Z}} \big(2^{qs} \|u_q\|_{L^p}\big)^r\Big)^{\frac{1}{r}}. \tag{2}$$

Proof: Under the hypothesis of the first assertion and according to Bernstein lemma, we have $\|u_q\|_{L^\infty} \leq C 2^{q\left(\frac{N}{p}-s\right)}$.

Thus implies that $\sum_q u_q$ is a convergent series in $\mathcal{S}'$. Next, we notice that there exists an integer N_0 so that

$$|q' - q| \geq N_0 \Longrightarrow \Delta_{q'} u_q = 0.$$

Therefore, with the convention that $u_q = 0$ if $q \leq -2$, we can write that

$$\|\Delta_{q'} u\|_{L^p} = \| \sum_{|q-q'|<N_0} \Delta_{q'} u_q \|_{L^p} \leq C \sum_{|q-q'|<N_0} \|u_q\|_{L^p}.$$

So, we obtain that

$$2^{q's} \|\Delta_{q'} u\|_{L^p} \leq C \sum_{|q-q'|\leq N_0} 2^{q's} \|u_q\|_{L^p} \leq C^{1+|s|} \sum_{|q-q'|\leq N_0} 2^{qs} \|u_q\|_{L^p},$$

and we deduce from convolution inequalities that

$$\|u\|_{B^s_{p,r}} \leq C^{1+|s|} \Big(\sum_{q\in\mathbb{Z}} 2^{rqs} \|u_q\|^r_{L^p} \Big)^{\frac{1}{r}},$$

whence the desired result. ■

The sequence $(\mathcal{F}(S_{p-1}u\Delta_p v)$ is supported in dyadic shells. Hence, it suffices to prove that:

$$(\sum_p (2^{qs}\|S_{p-1}u\Delta_p v\|_{L^2})^r)^{\frac{1}{r}} \lesssim \|u\|_{L^\infty}\|v\|_{B^s_{p,r}}.$$

Since $\|S_{q-1}u\|_{L^\infty} \leq \|\widetilde{h}\|_{L^1}\|u\|_{L^\infty}$, this is actually straightforward.
For proving the second result, we use that, because $\sigma > 0$, we have:

$$2^{q(s-\sigma)}\|S_{q-1}u\Delta_q v\|_{L^p} \leq (2^{qs}\|\Delta_q v\|_{L^p}) \sum_{q'\leq q-2} 2^{\sigma(q'-q)}2^{-q'\sigma}\|\Delta_{q'}u\|_{L^\infty}.$$

Therefore combining Hölder and convolution inequalities for series, we get:

$$(\sum_{q\in\mathbb{Z}}(2^{q(s-\sigma)}\|S_{q-1}u\Delta_q v\|_{L^p})^r)^{\frac{1}{r}} \leq (\sum_{k\geq 2} 2^{-k\sigma})\|u\|_{B^{-\sigma}_{\infty,r_1}}\|v\|_{B^s_{p,r}},$$

whence the desired inequality. ∎

To simplify the notation, we assume from now on that $\bar{\rho}=1$. Hence as long as ρ does not vanish, the equations for $(a=\rho^{-1}-1, u)$ read:

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0, \\ \partial_t u + u\cdot\nabla u + (1+a)(\nabla\Pi - \mu\Delta u) = f, \\ \operatorname{div} u = 0, \quad (a,u)_{/t=0} = (a_0, u_0). \end{cases} \tag{3}$$

Theorem (BH)

Let $1\leq r<\infty$, $1\leq p_1<\infty$ *and* $\varepsilon>0$ *such that:* $\frac{N}{p_1}+\varepsilon<\frac{N}{2}+1$ *and* $\frac{N}{2}\leq 1+\frac{N}{p_1}$.
Assume that $u_0\in B^{\frac{N}{2}-1}_{2,r}$ *with* $\operatorname{div} u_0=0$, $f\in\widetilde{L}^1_{loc}(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,r})$ *and* $a_0\in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}\cap L^\infty$, *with* $1+a_0$ *bounded away from zero. There exists a positive time* T *such that system (3) has a solution* (a,u) *with* $1+a$ *bounded away from zero and:*

$$a\in\widetilde{C}([0,T], B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}),\quad u\in\big(\widetilde{C}([0,T]; B^{\frac{N}{2}-1}_{2,r})\cap\widetilde{L}^1(0,T,B^{\frac{N}{2}+1}_{2,r})\big)^N$$

$$\text{and}\quad \nabla\Pi\in\widetilde{L}^1(0,T,B^{\frac{N}{2}-1}_{2,r}).$$

This solution is unique when $\frac{2}{N}\leq\frac{1}{p_1}+\frac{1}{2}$.

Parabolic equation: Let us first state estimates for the following constant coefficient parabolic system:

$$\begin{cases} \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = f, u_{/t=0} = u_0. \end{cases} \tag{4}$$

Proposition

Assume that $\mu \geq 0$ *and that* $\lambda + 2\mu \geq 0$ *with* $\nu = \min(\mu, \lambda+2\mu)$. *Then there exists a universal constant* κ *such that for all* $s \in \mathbb{Z}$ *and* $T \in \mathbb{R}^+$,

$$\|u\|_{\widetilde{L}^\infty_T(B^s_{p_1,1})} \leq \|u_0\|_{B^s_{p_1,1}} + \|f\|_{L^1_T(B^s_{p_1,1})},$$

$$\kappa\nu\|u\|_{L^1_T(B^{s+2}_{p_1,1})} \leq \sum_{l\in\mathbb{Z}} 2^{ls}(1-e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^{p_1}} + \|\Delta_l f\|_{L^1_T(L^{p_1})}),$$

Proof: Equation (4) has a unique solution u in $\mathcal{S}(0,T,\mathbb{R}^N)$ which satisfies:

$$\widehat{u}(t,\xi) = e^{-\mu t|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-\mu(t-\tau)|\xi|^2}\widehat{f}(\tau,\xi)d\tau.$$

Next we notice that applying Δ_q to (4) and using the previous formula yields:

$$\Delta_q u(t) = e^{\mu t\Delta}u_0 + \int_0^t e^{\mu(t-\tau)\Delta} f(\tau)d\tau.$$

Therefore: $\|\Delta_q u(t)\|_{L^p} = \|e^{\mu t\Delta}\Delta_q u_0\|_{L^p} + \int_0^t \|e^{\mu(t-\tau)\Delta}\Delta_q f(\tau)\|_{L^p} d\tau.$

Lemma

Let ϕ be a smooth function supported in the shell $\mathcal{C}(0, R_1, R_2)$ with $0 < R_1 < R_2$. There exist two positive constants κ and C depending only on ϕ and such that for all $1 \leq p \leq \infty$, $\tau \geq 0$ and $\lambda > 0$, we have

$$\|\phi(\lambda^{-1}D)e^{\tau\Delta}u\|_{L^p} \leq Ce^{-\kappa\tau\lambda^2}\|\phi(\lambda^{-1}D)u\|_{L^p}.$$

Performing a change of variable, one can assume with no loss of generality that $\lambda = 1$. Now, let $\widetilde{\phi}$ be a smooth function supported in $\mathcal{C}(0, R_1', R_2')$ for some $R_1' < R_1$ and $R_2' > R_2$ and such that $\widetilde{\phi} \equiv 1$ in a neighborhood of $\mathcal{C}(0, R_1, R_2)$. We have

$$\mathcal{F}\Big(\phi(D)e^{\tau\Delta}u\Big)(\xi) = \Big(\widetilde{\phi}(\xi)e^{-\tau|\xi|^2}\Big)\,\mathcal{F}(\phi(D)u)(\xi).$$

Thus, $\phi(D)e^{\tau\Delta}u = k_\tau \star \phi(D)u$ with

$$k_\tau(z) := (2\pi)^{-N}\int_{\mathbb{R}^N} e^{-\tau|\xi|^2}e^{iz\cdot\xi}\widetilde{\phi}(\xi)\,d\xi.$$

According to convolution inequalities, we have

$$\|\phi(D)e^{\tau\Delta}u\|_{L^p} \leq \|k_\tau\|_{L^1}\|\phi(D)u\|_{L^p}.$$

Therefore it only remains to prove that there exist two positive constants κ and C such that

$$\forall \tau \in \mathbb{R}^+, \|k_\tau\|_{L^1} \leq Ce^{-\kappa\tau}. \tag{5}$$

For that, we use the fact that for all $m \in \mathbb{N}$, we have

$$\begin{aligned}(1+|z|^2)^m k_\tau(z) &= (2\pi)^{-N} \int e^{-\tau|\xi|^2} \widetilde{\phi}(\xi)\,(Id - \Delta_\xi)^m \left(e^{iz\cdot\xi}\right) d\xi,\\ &= (2\pi)^{-N} \int e^{iz\cdot\xi}\,(Id - \Delta_\xi)^m \left(e^{-\tau|\xi|^2} \widetilde{\phi}(\xi)\,\right) d\xi.\end{aligned}$$

From the last equality and the fact that the integration may be restricted to the shell $\mathcal{C}(0, R_1', R_2')$, we easily conclude that there exists a constant C_m such that

$$\forall z \in \mathbb{R}^N,\ (1+|z|^2)^m |k_\tau(z)| \leq C_m e^{-\kappa\tau},$$

whence inequality (5). ■

By virtue of lemma, we thus have for some $\kappa > 0$:

$$\begin{aligned}\|\Delta_q u(t)\|_{L^{\rho_1}_T(L^p)} \lesssim \Big(\frac{1 - e^{-\kappa\mu T\rho_1 2^{2q}}}{\kappa\mu\rho_1 2^{2q}}\Big)^{\frac{1}{\rho_1}} \|\Delta_q u_0\|_{L^p}\\ + \Big(\frac{1 - e^{-\kappa\mu T\rho_2 2^{2q}}}{\kappa\mu\rho_1 2^{2q}}\Big)^{\frac{1}{\rho_2}} \|\Delta_q f\|_{L^\rho_T(L^p)},\end{aligned}$$

with $\frac{1}{\rho_2} = 1 + \frac{1}{\rho_1} - \frac{1}{\rho}$. Finally taking the $l^r(\mathbb{Z})$ norm, we conclude.

Idea of the proof for (INS): The main result is the following:

Theorem

Let $1 \leq r \leq \infty$ and $1 \leq p < \infty$. There exists a constant $c > 0$ independent of μ such that for all divergence free vector-field u_0 with coefficients in $\dot{B}^{\frac{N}{p}-1}_{p,r}$ and external force f with coefficients in $\widetilde{L}^1(\dot{B}^{\frac{N}{p}-1}_{p,r})$ such that

$$\|u_0\|_{B^{\frac{N}{p}-1}_{p,r}} + \|\mathcal{P}f\|_{\widetilde{L}^1(B^{\frac{N}{p}-1}_{p,r})} < c\mu, \tag{6}$$

system (NS) has a unique solution u in $\widetilde{L}^1(\dot{B}^{\frac{N}{p}+1}_{p,r}) \cap \widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,r})$ which satisfies

$$\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,r})u + \mu\widetilde{L}^1(\dot{B}^{\frac{N}{p}+1}_{p,r})u < 2c\mu. \tag{7}$$

Besides, if r is finite then u belongs to $\mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{N}{p}-1}_{p,r})$ and uniqueness holds true in $\widetilde{L}^1(\dot{B}^{\frac{N}{p}+1}_{p,r}) \cap \widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,r})$ with no smallness condition.

The proof of theorem is based on the following lemma.

Lemma

Let $(X,\|\cdot\|_X)$ be a Banach space and $B : X\times X\to X$ be a bilinear continuous operator with norm K. Then for all $y\in X$ such that $4K\|y\|_X<1$, equation $x=y+B(x,x)$ has a unique solution x in the ball $\mathcal{B}(0,\frac{1}{2K})$. Besides x satisfies $\|x\|_X\leq 2\|y\|_X$.

Let us now prove the result. The existence of a solution with data u_0 and f will be obtained by applying the previous lemma for convenient y, B and $(X,\|\cdot\|_X)$. For X, we shall take the space of divergence free distributions over $\mathbb{R}^+\times\mathbb{R}^N$ with coefficients in $\widetilde{L}^1(\dot{B}^{\frac{N}{p}+1}_{p,r})\cap\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,r})$ endowed with the norm

$$\|v\|_X:=\widetilde{L}^\infty(\dot{B}^{\frac{N}{p}-1}_{p,r})v+\mu\widetilde{L}^1(\dot{B}^{\frac{N}{p}+1}_{p,r})v.$$

We then set $y: t\mapsto e^{\mu t\Delta}u_0+\int_0^t e^{\mu(t-\tau)\Delta}\mathcal{P}f\,d\tau$ and define the bilinear functional B by the formula

$$B(v,w)(t)=-\int_0^t e^{\mu(t-\tau)\Delta}\mathcal{P}\mathrm{div}\,(v(\tau)\otimes w(\tau))\,d\tau.$$

We claim that y belongs to X, that B maps $X\times X$ in X and that there exists some constant C such that

$$\|y\|_X\leq C\big(\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,r}}+\|\mathcal{P}f\|_{\widetilde{L}^1(\dot{B}^{\frac{N}{p}-1}_{p,r})}\big), \tag{8}$$

$$\forall (v,w) \in X^2,\ \|B(v,w)\|_X \leq C\mu^{-1}\|v\|_X\|w\|_X. \tag{9}$$

Indeed, as u_0 is divergence free and belongs to $\dot{B}^{\frac{N}{p}-1}_{p,r}$, and as f is in $\widetilde{L}^1(\dot{B}^{\frac{N}{p}-1}_{p,r})$, the proposition on the heat equation insures that y belongs to X and satisfies (8). Next, using Bony's decomposition and $\operatorname{div} v = \operatorname{div} w = 0$, one can write

$$\operatorname{div}(v \otimes w) = \dot{T}_{\partial_j v} w^j + \dot{T}_{w^j}\partial_j v + \partial_j \dot{R}(v, w^j)$$

with the summation convention over repeated indices. Hence, combining proposition on the paraproduct and the embedding $\widetilde{L}^\rho(B^{\frac{N}{p}-\frac{1}{2}}_{p,r}) \hookrightarrow \widetilde{L}^\rho(B^{-\frac{1}{2}}_{\infty,\infty})$ for $\rho = 4/3$ or $\rho = 4$, we get

$$\|\operatorname{div}(v\otimes w)\|_{\widetilde{L}^1(B^{\frac{N}{p}-1}_{p,r})} \leq C\|v\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{p}+\frac{1}{2}}_{p,r})}\|w\|_{\widetilde{L}^4(B^{\frac{N}{p}-\frac{1}{2}}_{p,r})}.$$

whence by interpolation:

$$\|\operatorname{div}(v\otimes w)\|_{\widetilde{L}^1(B^{\frac{N}{p}-1}_{p,r})} \leq C\mu^{-1}\|v\|_X\|w\|_X.$$

Finally, by using the fact that $\mathcal{P}$ is an homogeneous multiplier of degree 0, and by applying proposition on heat equation $B(v,w)$ belongs to X and that (9) is satisfied.

Now, the lemma may be applied provided that $4C\|y\|_X < \mu$. According to (8) this condition will be satisfied if

$$\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,r}} + \|\mathcal{P}f\|_{\widetilde{L}^1(\dot{B}^{\frac{N}{p}-1}_{p,r})} < c\mu$$

for some small enough constant c.
This achieves the proof of existence of a global solution in X for (INS), and of uniqueness under condition (7). ■

Next we have to study the linearization of the momentum equation:

$$\begin{cases} \partial_t u + b(\nabla\Pi - \mu\Delta u) = g, \\ \operatorname{div} u = 0, u_{/t=0} = u_0. \end{cases} \tag{10}$$

where b, g and u_0 are given. We assume that $u_0 \in H^s$ and $f \in \widetilde{L}^1(0,T;B^s_{p,r})$, that b is bounded by below by a positive constant $\underline{b}$ and that $a = b - 1$ belongs to $L^\infty(0,T;B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty)$.

Let us introduce the following notation:

$$\mathcal{A}_T = 1 + \underline{b}^{-1}\|\nabla b\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha-1}_{p_1,\infty})} \quad \text{if } \alpha \neq 1. \tag{11}$$

Proposition

Let $\underline{\nu} = \underline{b}\mu$ and $(p, p_1) \in [1, +\infty]$. Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies: $\inf_{(t,x)\in[0,T)\times\mathbb{R}^N} b_m(t, x) \geq \frac{b}{2}$. There exist three constants c, C and κ (with c, C, depending only on N and on s, and κ universal) such that if in addition we have:

$$\|a - S_m a\|_{L^\infty(0,T;B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leq c\frac{\underline{\nu}}{\mu} \tag{12}$$

then setting: $Z_m(t) = 2^{2m}\mu^2\underline{\nu}^{-1}\int_0^t \|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau$, We have for all $t \in [0, T]$

with $\alpha' > 0$,

$$\|u\|_{\widetilde{L}^\infty_T(H^s)} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T(H^{s+2})} \leq e^{CZ_m(T)}\big(\|u_0\|_{H^s} + \big(\frac{\underline{\nu}(p-1)}{p}\big)\mathcal{A}^\kappa_T(\|\mathcal{P}g\|_{\widetilde{L}^1_T(H^s)} + \|u\|_{\widetilde{L}^1_T(H^{s+2-\alpha'})})\big). \tag{13}$$

Let us first rewrite (10) as follows:

$$\begin{cases} \partial_t u - b_m \mu \Delta u + b \nabla \Pi = f + g + E_m, \\ \mathrm{div} u = 0, u_{t=0} = u_0. \end{cases} \tag{14}$$

with $E_m = \mu \Delta u \,(\mathrm{Id} - S_m) a$ and $b_m = 1 + S_m a$. Note that:

$$\|E_m\|_{B^s_{p,r}} \lesssim \|a - S_m a\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} \|D^2 u\|_{B^s_{p,r}}. \tag{15}$$

Now applying operator Δ_q and next operator of free divergence yield $\mathcal{P}$ to momentum equation (14) yields:

$$\frac{d}{dt} u_q - \mu \mathrm{div}(b_m \nabla u_q) = \mathcal{P} g_q + \Delta_q \mathcal{P} E_m + \widetilde{R}_q - \Delta_q \mathcal{P}(a \nabla \Pi), \tag{16}$$

with:

$$\widetilde{R}_q = \mu\big(\mathcal{P}\Delta_q(b_m \Delta u) - \mathcal{P}\mathrm{div}(b_m \nabla u_q)\big) - \mu \mathcal{Q}\mathrm{div}(S_m a \nabla u_q).$$

Next multiplying both sides by $|u_q|^{p-2} u_q$, and integrating we get:

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u_q\|^p_{L^p} + \mu \int_{\mathbb{R}^N} b_m |\nabla u_q|^2 |u_q|^{p-2} dx \\ &+ \mu \int_{\mathbb{R}^N} b_m |u_q|^{p-4} |\nabla |u|^2|^2 dx \leq \|u_q\|^{p-1}_{L^p} (\|\mathcal{P} g_q\|_{L^p} + \|\widetilde{R}_q\|_{L^p} \\ &+ \|\Delta_q(T_{\nabla a} \nabla \Pi)\|_{L^p} + 2^q \|\Delta_q(T_a \Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla \Pi} a)\|_{L^p} \\ &\qquad + \|\mathcal{P}\Delta_q E_m\|_{L^p}). \end{aligned}$$

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0, \\ a_{/t=0} = a_0. \end{cases} \tag{17}$$

In order to measure precisely the regularity of the vector field u, we shall introduce the following notation:

$$V'_{p_2,\alpha}(t) = \sup_{j\geq 0} \frac{2^{j\frac{N}{p_2}}\|\nabla S_j u(t)\|_{L^{p_2}}}{(j+1)^\alpha} < +\infty. \tag{18}$$

Theorem

Let (p, p_2) *be in* $[1,+\infty]^2$ *such that:*

- *if* $\frac{1}{p}+\frac{1}{p_2}\leq 1$ *then* $\sigma\in]-1-\frac{N}{p_2}, 1+N\inf(\frac{N}{p},\frac{N}{p_2})[$
- *if* $\frac{1}{p}+\frac{1}{p_2}\geq 1$ *then* $\sigma\in]-1-\frac{N}{p'_2}, 1+N\inf(\frac{N}{p},\frac{N}{p_2})[$

Assume that $V'_{p_2,\alpha}$ *with* $\alpha\in]0,1[$ *is in* $L^1([0,T])$. *Let* $a_0\in B^\sigma_{p,\infty}$, *the equation (17) has a unique solution* $a\in C([0,T],\cap_{\sigma'<\sigma}B^{\sigma'}_{p,\infty})$ *and the following estimate holds for all small enough* ε:

$$\|a\|_{\widetilde{L}^\infty_T(B^{\sigma-\varepsilon}_{p,\infty})} \leq C\|a_0\|_{B^\sigma_{p,\infty}} e^{\left(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_2,\alpha}(T))^{\frac{1}{1-\alpha}}\right)},$$

where C *depends only on* α, p, p_1, σ *and* N.

Proof of theorem [BH]:

We smooth on the data as follows:

$$a_0^n = S_n a_0,\quad u_0^n = S_n u_0 \quad \text{and} \quad f^n = S_n f.$$

Now, according H. Abidi, R. Danchin and M. Paicu, one can solve (3) with the smooth data (a_0^n, u_0^n, f^n). We get a solution (a^n, u^n) on a non trivial time interval $[0, T_n]$.

Uniform bounds:

Let T_n be the lifespan of (a_n, u_n), that is the supremum of all $T>0$ such that (3) with initial data (a_0^n, u_0^n) has a solution. Let T be in $(0, T_n)$. We aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution $(u_L^n, \nabla\Pi_L^n)$ to the nonstationary Stokes system:

$$(L)\quad \begin{cases} \partial_t u_L^n - \mu\Delta u_L^n + \nabla\Pi_L^n = f^n, \\ \operatorname{div} u_L^n = 0, \\ (u_L^n)_{t=0} = u_0^n. \end{cases}$$

Now, the vectorfields $\widetilde{u}^n = u^n - u_L^n$ and $\nabla\Pi^n = \nabla\Pi_L^n + \nabla\widetilde{\Pi}^n$ satisfy the parabolic system:

$$\begin{cases} \partial_t \widetilde{u}^n - \mu(1+a^n)\Delta \widetilde{u}^n + (1+a^n)\nabla \widetilde{\Pi}^n = H(a^n, u^n, \nabla \Pi^n), \\ \mathrm{div}\widetilde{u}^n = 0, \widetilde{u}^n(0) = 0, \end{cases} \tag{19}$$

Define $m \in \mathbb{Z}$ by:

$$m = \inf\{p \in \mathbb{Z}/ \ \ 2\bar{\nu} \sum_{l \geq q} 2^{l\frac{N}{p_1}} \|\Delta_l a_0\|_{L^{p_1}} \leq c\bar{\nu}\} \tag{20}$$

where c is small enough positive constant. We can show for some $\eta > 0$:

$(\mathcal{H}_1)$ $\quad \|a^n - S_m a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty}) \cap L^\infty} \leq c\underline{\nu}\bar{\nu}^{-1},$

$(\mathcal{H}_2)$ $\quad C\bar{\nu}^2 T \|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty}) \cap L^\infty} \leq 2^{-2m}\underline{\nu},$

$(\mathcal{H}_3)$ $\quad \frac{1}{2}\underline{b} \leq 1 + a^n(t,x) \leq 2\bar{b}$ for all $(t,x) \in [0,T] \times \mathbb{R}^N,$

$(\mathcal{H}_4)$ $\quad \|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}) \cap L^\infty} \leq A_0,$

$(\mathcal{H}_5)$ $\quad \|u^n_L\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq \eta,$

$(\mathcal{H}_6)$ $\quad \|\widetilde{u}^n\|_{\widetilde{L}^\infty(B^{\frac{N}{p_2}-1}_{p_2,r})} + \underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq \widetilde{U}_0\eta,$

By a argument of blow-up we show that $\forall n \in \mathbb{N}$, we have $T \leq T_n$. The proof is based on Ascoli's theorem and compact embedding for Besov spaces.

Uniqueness when $1 \leq p_2 < 2N$, $\frac{2}{N} < \frac{1}{p_1} + \frac{1}{p_2}$ and $N \geq 3$.:

We assume that we are given two solutions (a^1, u^1, Π_1) and $(a^2, u^2, , \Pi_2)$. Let $\delta a = a^2 - a^1$, $\delta u = u^2 - u^1$ and $\delta\Pi = \Pi^2 - \Pi^1$. The system for $(\delta a, \delta u, \delta\Pi)$ reads:

$$\begin{cases} \partial_t \delta a + u^2 \cdot \nabla \delta a = -\delta u \cdot \nabla a^1, \\ \partial_t \delta u + u^2 \cdot \nabla \delta u + \delta u \cdot \nabla u^1 - (1 + a^1)\Delta \delta u + \nabla \delta\Pi \\ \qquad\qquad\qquad\qquad\qquad\qquad = F(a, u, \Pi). \end{cases} \tag{21}$$

with:

$$F(a, u, \Pi) = -\delta u \cdot \nabla u^1 + a^1(\mu\Delta\delta u - \nabla\delta\Pi) + \delta a(\mu\Delta u^2 - \nabla\Pi^2).$$

Due to the term $\delta u \cdot \nabla a^1$ in the right-hand side of the first equation, we loose one derivative when estimating δq. Now, the right hand-side of the second equation contains a term of type $\delta a \mu \Delta u^2$ so that the loss of one derivative for δq entails a loss of one derivative for δu. Therefore, we need to get bounds on $(\delta a, \delta u, \delta\Pi)$ in:

$$\begin{aligned} F_T = & \widetilde{C}_T(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}) \times \big(\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,r}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,r})\big)^N \\ & \times \widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r}) \end{aligned}$$

We next estimate $(\delta a, \delta u, \delta\Pi)$ by theorem 9 and proposition 1.5, and have to apply Grönwall lemma.

Uniqueness when $p_2 = 2N$ or $N = 2$.

In the present case, the above heuristic fails because we have reached some limit cases for the product laws. Indeed, a term such as $\delta u \cdot \nabla u_1$ cannot be estimated properly: the product does not map $B^{\frac{N}{p_2}-1}_{p_2,r} \times B^{\frac{N}{p_2}}_{p_2,r}$.
The key to that difficulty relies on the following logarithmic interpolation inequality:

$$\|u\|_{L^1_T(B^1_{N,1})} \lesssim \|u\|_{\widetilde{L}^1_T(B^1_{N,\infty})} \log\Big(e + \frac{\|u\|_{\widetilde{L}^1_T(B^{1+\varepsilon}_{N,\infty})}}{\|u\|_{\widetilde{L}^1_T(B^1_{N,\infty})}}\Big),$$

Therefore, getting bounds on $(\delta a, \delta u, \delta\Pi)$ in:

$$\begin{aligned} F_T =&\widetilde{C}_T(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}) \times \big(\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})\big)^N \\ &\times \widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,\infty}) \end{aligned}$$

and a well-known generalization of Grönwall the Osgood's lemma.

Perspectives:

- 1. Improving the hypothesis for the uniqueness (in collaboration with P. Germain, Courant Institut).
- 2. Existence of self similar solution (in collaboration with P. Germain).
- 3. Ill-posedness in the space critical for the scaling and with $u_0 \in B_{2,\infty}^{\frac{N}{2}-1}$ (in collaboration with P. Germain).

- Mass conservation :

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0.$$

- Momentum conservation :

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \operatorname{div} \mathbf{D}.$$

- Energy conservation :

$$\partial_t\big(\rho(e + \tfrac{1}{2}|\mathbf{u}|^2)\big) + \operatorname{div}\big[\big(\rho(e + \tfrac{1}{2}|\mathbf{u}|^2) + p_0\big)\mathbf{u}\big] = \operatorname{div}\big((\mathbf{D} - \mathbf{Q} + \mathbf{W}\big).$$

- Viscous tensor : $\mathbf{D} := (\lambda \operatorname{div} \mathbf{u})\mathbf{I}_N + \mu(D\mathbf{u} + \nabla \mathbf{u})$.
- Thermal conductivity : $\mathbf{Q} = -\eta \nabla T$.
- Interstitial working : $\mathbf{W} := -\rho\kappa\nabla\rho \operatorname{div} \mathbf{u}$.

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density. The pressure P is a suitable smooth function of ρ. We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ (in the sequel to simplify the calculus we will assume the viscosity coefficients are constant functions). Such a conditions ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$. We supplement the problem with initial condition (ρ_0, u_0, e_0).

The incompressible Navier-Stokes system equations reads:

$$(INS)\begin{cases}\partial_t \mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta u+\nabla\Pi=0\\ \mathrm{div}u=0\end{cases}$$

with Π the pressure and u the velocity.

Energy inequality and existence of global weak solutions :

We have the following energy inequalities:

$$\int_{\mathbb{R}^N}\frac{1}{2}|u|^2(t,x)dx+\int_0^t\int_{\mathbb{R}^N}\mu|\nabla u|^2dxdt\leq\int_{\mathbb{R}^N}\frac{1}{2}|u_0|^2dx,$$

- Existence of global weak solution for $N\geq 2$, Leray (1930),

The proof consists in building approximated solutions and pass to the limit (for this we use the Sobolev injection).

Energy inequality :

Before tackling the global stability theory for the system (CNS), let us derive formally the uniform bounds available on (ρ, u). Let Π (free energy) be defined by:

$$\Pi(s) = s\Big(\int_0^s \frac{P(z)}{z^2} dz\Big), \tag{22}$$

so that $P(s) = s\Pi^{'}(s) - \Pi(s)$, $\Pi^{'}(\bar{\rho}) > 0$ and if we renormalize the mass equation:

$$\partial_t \Pi(\rho) + \text{div}(u\Pi(\rho)) + P(\rho)\text{div}(u) = 0 \;\text{ in }\; \mathcal{D}^{'}((0,T)\times\mathbb{R}^N).$$

Notice that Π is convex whenever P is nondecreasing. Multiplying the equation of momentum conservation by u and integrating by parts over $\mathbb{R}^N$, we obtain the following energy estimate:

$$\begin{aligned}\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho|u|^2 + \Pi(\rho)\big)(t,x)dx + \int_0^t\int_{\mathbb{R}^N}(\mu D(u) : D(u)\\ + (\lambda+\mu)|\text{div} u|^2)dx \leq \int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho} + \Pi(\rho_0)\big)dx,\end{aligned} \tag{23}$$

We are interested in using the above inequality energy to determine the functional space we must work with. In view of (23), we can specify initial conditions on $\rho_{/t=0} = \rho_0$ and $\rho u_{/t=0} = m_0$ where we assume that:

$$\begin{aligned} &\bullet\ \rho_0 \geq 0 \ \text{a.e in } \mathbb{R}^N,\ \rho_0 \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N) \ \text{ with } s = \max(2,\gamma), \\ &\bullet\ m_0 = 0 \ \text{a.e on } \rho_0 = 0, \\ &\bullet\ \frac{|m_0|^2}{\rho_0} \ \text{(defined to be 0 on } \rho_0 = 0) \text{ is in } L^1(\mathbb{R}^N). \end{aligned} \tag{24}$$

We deduce the following a priori bounds which give us the energy space in which we will work:

- $\rho \in L^\infty(0,T; L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N))$,
- $\rho|u|^2 \in L^\infty(0,T; L^1(\mathbb{R}^N))$,
- $\nabla u \in L^2((0,T) \times \mathbb{R}^N)^N$.

We wish to prove global stability results for (CNS) in functional spaces very close to energy spaces. In the barotropic case $P(\rho) = a\rho^\gamma$, P-L. Lions proved the global existence of weak solutions (ρ, u) to (CNS) for initial data in the energy spaces. In this case, assuming that have built approximated solutions of the system (ρ_n, u_n), we get by energy inequalities:

$$P(\rho_n) \text{ is uniformly bounded in } L^\infty(L^1)$$

It means that $P(\rho_n)$ converges in the sense of the measure to a measure $\bar{\mu}$. The question is to know if:

$$P(\rho) = \bar{\mu}?$$

Roughly the proof consists in two steps:

- First step: We build global approximated solutions of the system. To do this, we generally add some regularizing terms of viscous type whit some coefficients ε_n which tends to 0 when n tends to infinity. The main difficulty is to keep the uniform energy bounds for the solutions (ρ_n, u_n).
- Second step: We check that the approximated solutions converge in the sense of the distribution to a solution of our system. It requires some properties of compactness.

In the sequel we will assume always construct by a mollifying processus a sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of global approximated solutions for the system (CNS). The sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ are regular bounded energy weak solutions (it means that they check uniformly the energy bound) , more precisely they verify the following conditions:

- $\rho_n \in L^\infty(0,T;L^1(\mathbb{R}^N)\cap L^s(\mathbb{R}^N))$,
- $\rho_n|u_n|^2 \in L^\infty(0,T;L^1(\mathbb{R}^N))$,
- $\nabla u_n \in L^2((0,T)\times\mathbb{R}^N)^N$.

in addition, we assume that the sequence ρ_n is bounded in $L^r((0,T)\times\mathbb{R}^N))\cap L^1((0,T)\times\mathbb{R}^N)$ for some $r>\gamma$. In the proof, we will explain this extra condition on the sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$. Indeed via some energy methods we can get a gain of integrability on the density.

Theorem (P-L Lions, 96)

Let $\gamma > N/2$ *if* $N \geq 4$, $\gamma \geq \frac{9}{5}$ *if* $N \geq 3$ *and* $\gamma \geq \frac{3}{2}$ *if* $N \geq 2$.
Let the couple (ρ_0^n, u_0^n) *satisfy:*

- ρ_n^0 *is uniformly bounded in* $L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ *with* $s = \max(\gamma, 2)$ *and* $\rho_0^n \geq 0$ *a.e in* $\mathbb{R}^N$,
- $\rho_n^0 |u_0^n|^2$ *is uniformly bounded in* $L^1(\mathbb{R}^N)$,
- $\rho_0^n u_0^n = 0$ *whenever* $x \in \{\rho_0 = 0\}$.

In addition we suppose that ρ_n^0 *converges in* $L^1(\mathbb{R}^N)$ *to* ρ_0. *Then, up to a subsequence,* (ρ_n, u_n) *converges strongly to a weak solution* (ρ, u) *of the system* (NSK) *satisfying the initial condition* (ρ_0, u_0) *as in (24). Moreover we have the following convergence:*

- $\rho_n \to_n \rho$ *in* $C([0,T], L^p(\mathbb{R}^N)) \cap L^r((0,T) \times \mathbb{R}^N)$ *for all* $1 \leq p < s$, $1 \leq r < q$, *with* $q = s + \frac{N\gamma}{2} - 1$ *if* $N = 3$.
- $\rho_n \to_n \rho$ *in* $C([0,T], L^p(\mathbb{R}^2)) \cap L^r((0,T) \times K)$ *for all* $1 \leq p < s$, $1 \leq r < q$, *with* K *an arbitrary compact in* $\mathbb{R}^2$ *if* $N = 2$.

In addition we have:

- $\rho_n u_n \to \rho u$ *in* $L^p(0,T; L^r(\mathbb{R}^N))$ *for all* $1 \leq p < +\infty$ *and* $1 \leq r < \frac{2s}{s+1}$,

We can check easily that with the previous theorem, we have shown that if we have built a sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of global approximate solutions verifying the properti

Sketch of the proof:

First step: Gain of integrability on the density

The energy inequalities give us just a control $L^\infty(L^1)$ on the pressure terms $P(\rho_n)$, in the goal to test this pressure terms with convex function, we need to obtain a gain of integrability on this term. To do this, we have the following proposition:

Proposition

Let $N = 2,3$ and $\gamma \geq 1$. Let (ρ, u) be a regular bounded energy weak solution of the system (NSK) with $\rho \geq 0$ and $\rho \in L^\infty(L^1 \cap L^{s+\varepsilon})$ where we define ε below.

$$\int_{(0,T)\times\mathbb{R}^N} (\rho^{\gamma+\varepsilon} + \rho^{2+\varepsilon})dxdt \leq M \text{ for any } 0 < \varepsilon \leq \frac{2}{N}\gamma - 1.$$

with M depending only on the initial conditions and on the time T.

We will present only the case $N = 3$.
Applying the operator $(-\Delta)^{-1}\text{div}$ to the momentum equation yields:

$$a\rho^\gamma = \frac{\partial}{\partial t}(-\Delta)^{-1}\text{div}(\rho u) + (-\Delta)^{-1}\partial^2_{i,j}(\rho u_i u_j) + (2\mu + \lambda)\text{div}u. \tag{25}$$

so that multiplying by ρ^ε with $0<\varepsilon\leq\min(\frac{1}{N},\frac{2}{N}\gamma-1)$, we get:

$$a\rho^{\gamma+\varepsilon}=\rho^\varepsilon(-\Delta)^{-1}\partial^2_{ij}(\rho(u_i)(u_j))+\frac{\partial}{\partial t}\big(\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\big)+\mathrm{div}[u\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)]$$
$$+(\mu+2\lambda)\mathrm{div}u-(\rho)^\varepsilon u\cdot\nabla(-\Delta)^{-1}\mathrm{div}(\rho u)+(1-\varepsilon)(\mathrm{div}u)\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\ . \tag{26}$$

Next we integrate (26) in time on $[0,T]$ and in space. We get:

$$\int_{(0,T)\times\mathbb{R}^N}a\rho^{\gamma+\varepsilon}dx\,dt=\int_{(0,T)\times\mathbb{R}^N}\Big(\frac{\partial}{\partial t}[\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)]$$
$$+(\mu+\zeta)(\mathrm{div}u)\rho^\varepsilon+(1-\varepsilon)(\mathrm{div}u)\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)$$
$$+\rho^\varepsilon[R_iR_j(\rho u_iu_j)-u_iR_iR_j(\rho u_j)]+\mathrm{div}[u\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)]\Big)dx\,dt\ , \tag{27}$$

where R_i is the classical Riesz transform. Now we want to control the term $\int_0^T\int_{\mathbb{R}^N}(\rho^{\gamma+\varepsilon})dxdt$. As ρ is positive, it will enable us to control $\|\rho\|_{L^{\gamma+\varepsilon}_{t,x}}$. This may be achieved by bounding each term on the right side of (27).

Second step: Compactness results

The idea of the proof will be to test the convergence of the sequence $(\rho_n)_{n\in\mathbb{N}}$ on concave functions B in order to use their properties of lower semi-continuity with respect to the weak topology in $L^1(\mathbb{R}^N)$.

First, we can rewrite mass conservation of the regular solution $(\rho_n, u_n)_{n\in\mathbb{N}}$ on the form:

$$\frac{\partial}{\partial t}(B(\rho_n)) + \mathrm{div}(u_n B(\rho_n)) = (B(\rho_n) - \rho_n B^{'}(\rho_n))\mathrm{div} u_n,$$

where $B(\rho) = \rho^{\varepsilon}$. Supposing that $B(\rho_n)$ is bounded in appropriate space we can pass to the weak limit where we have in the energy space $\rho_n \rightharpoonup \rho$ and $u_n \rightharpoonup u$, so we get:

$$\frac{\partial}{\partial t}(\overline{B(\rho)}) + \mathrm{div}(u\overline{B(\rho)}) = \overline{b(\rho)\mathrm{div} u} \quad \text{with} \;\; b(\rho) = (1-\varepsilon)\rho^{\varepsilon}. \tag{28}$$

where in the sequel $\overline{B(\rho)}$ will represent the weak limit of $\overline{B(\rho_n)}$. Next by compactness arguments we can pass to the limit in the transport equation to obtain that:

$$\frac{\partial}{dt}\rho + \mathrm{div}(\rho u) = 0. \tag{29}$$

A crucial point now is to use the theory of the renormalized solutions introduced by Di Perna and Lions, indeed we want to compare ρ and $\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}$. Consequently, in order to estimate the difference $\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}} - \rho$ which tests the convergence of ρ_n, we need to estimate the difference $\overline{b(\rho)\mathrm{div}(u)} - \overline{b(\rho)}\mathrm{div}(u)$.

So now we aim at estimating the difference $\overline{b(\rho)\text{div}u} - b(\rho)\text{div}u$. This may be achieved by introducing the effective viscous pressure $\text{P}_{eff} = P - (2\mu+\lambda)\text{div}u$ after D. Hoff , which satisfies some important properties of weak convergence.

We obtain by two different way the following equalities:

$$\begin{aligned}
[\zeta\overline{\text{div}u\,\rho^{\varepsilon}} - \overline{(a\rho^{\gamma+\varepsilon})}] = &-\frac{\partial}{\partial t}[\overline{\rho^{\varepsilon}}(-\Delta)^{-1}\text{div}(\rho u)] - \text{div}[\overline{\rho^{\varepsilon}}u(-\Delta)^{-1}\text{div}(\rho u)] \\
&+\overline{\rho^{\varepsilon}}[u.\nabla(-\Delta)^{-1}\text{div}(\rho u) - (-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)] \\
&+(1-\varepsilon)\overline{\text{div}u\,\rho^{\varepsilon}}(-\Delta)^{-1}\text{div}(\rho u).
\end{aligned} \tag{30}$$

$$\begin{aligned}
[\zeta\text{div}u\,\overline{\rho^{\varepsilon}} - \overline{(a\rho^{\gamma})}\;\overline{\rho^{\varepsilon}})] = &\overline{\rho^{\varepsilon}}(-\Delta)^{-1}\text{div}(\overline{\rho(\nabla\phi*\rho)}) \\
&-\overline{\rho^{\varepsilon}}\frac{\partial}{\partial t}[\rho(-\Delta)^{-1}\text{div}(\rho u)] + \overline{\rho^{\varepsilon}}[u.\nabla(-\Delta)^{-1}\text{div}(\rho u) \\
&-(-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)] - \text{div}[\rho^{\varepsilon}u(-\Delta)^{-1}\text{div}(\rho u)] \\
&+(1-\varepsilon)\overline{\text{div}u(\rho)^{\varepsilon}}\,(-\Delta)^{-1}\text{div}(\rho u).
\end{aligned} \tag{31}$$

Subtracting (31) from (30), we get:

$$\zeta\,\overline{\mathrm{div}u(\rho)^{\varepsilon}} - a\overline{\rho^{\gamma+\varepsilon}} = \zeta\mathrm{div}u\,\overline{\rho^{\varepsilon}} - a\overline{\rho^{\gamma}}\,\overline{\rho^{\varepsilon}}\quad \text{a.e}\,.$$

Next we observe that by convexity:

$$(\overline{\rho^{\gamma+\varepsilon}})^{\frac{\varepsilon}{\gamma+\varepsilon}} \geq (\overline{\rho^{\varepsilon}}),\quad (\overline{\rho^{\gamma+\varepsilon}})^{\frac{\gamma}{\gamma+\varepsilon}} \geq (\overline{\rho^{\gamma}})\quad \text{a.e}\,.$$

So we get:

$$\overline{\mathrm{div}u\,(\rho)^{\varepsilon}} \geq \mathrm{div}u\,\overline{\rho^{\varepsilon}}. \tag{32}$$

From (28), we get:

$$\frac{\partial}{\partial t}\overline{\rho^{\varepsilon}} + \mathrm{div}(u\,\overline{\rho^{\varepsilon}}) \geq (1-\varepsilon)\overline{\rho^{\varepsilon}}\mathrm{div}u. \tag{33}$$

Next if ρ is L^2_{loc}, by the theory of renormalized solutions introduced by Di Perna and Lions we are able to prove that:

$$\frac{\partial}{\partial t}\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}} + \mathrm{div}(u\,\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}) \geq 0. \tag{34}$$

We set $r = \rho - \overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}$. and thus:

$$\frac{\partial}{\partial t}r + \mathrm{div}(ur) \leq 0. \tag{35}$$

We use now the well-known theorem.

Theorem

Let $(v_n)_{n\in\mathbb{N}}$ *be a sequence of functions bounded in* $L^1(\mathbb{R}^N)$ *such that:*

$$v_n \rightharpoonup v \quad \text{weakly in } L^1(\mathbb{R}^N).$$

Let $\varphi : \mathbb{R} \longrightarrow [-\infty, +\infty)$ *be a upper semi-continuous strictly concave function such that* $\varphi(v_n) \in L^1(\mathbb{R}^N)$ *for any* n *, and:*

$$\varphi(v_n) \rightharpoonup \overline{\varphi(v)} \quad \text{weakly in } L^1(\mathbb{R}^N).$$

Then:

$$\varphi(v) \geq \overline{\varphi(v)}.$$

and if $\varphi(v) = \overline{\varphi(v)}$ *then:*

$$v_n(y) \to v(y) \quad a.e.$$

extracting a subsequence as the case may be.

We have then $r \geq 0$ and by (35) $r = 0$, which conclude the proof.