Cole-Hopf transformation as numerical tool for the Burgers Equation

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Outline

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Cole-Hopf transformation

The Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

using the Cole-Hopf transformation, given by

$$u(x,t) = -2\nu\frac{\frac{\partial}{\partial x}\Theta(x,t)}{\Theta(x,t)}$$

is transformed into the linear diffusion equation

$$\frac{\partial \Theta}{\partial t} = \nu\frac{\partial^2 \Theta}{\partial x^2}$$

that, for the initial value problem, has the solution

$$\Theta(x,t) = \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty}\Theta(y,0)e^{-\frac{(y-x)^2}{4\nu t}}\,dy$$

Cole-Hopf transformation

Let us observe that

$$u(x,t) = -2\nu \frac{\frac{\partial}{\partial x}\Theta(x,t)}{\Theta(x,t)} \Longrightarrow \Theta(x,t) = e^{-\frac{1}{2\nu}\int_\alpha^x u(\xi,t)d\xi}$$

For convenience, we denote:

$$G(x,y) := e^{-\frac{(y-x)^2}{4\nu t}}$$

Then:

$$u(x,t) = -\frac{\int_{-\infty}^{\infty}(y-x)\Theta(y,0)G(x,y)dy}{t\int_{-\infty}^{\infty}\Theta(y,0)G(x,y)dy}$$

Outline

1. Cole-Hopf transformation

2. Numerical schemes

3. Some results

Numerical schemes

We have:

$$\Theta(x,t) = e^{-\frac{1}{2\nu}\int_{\alpha}^{x} u(\xi,t)d\xi}$$

$$u(x,t) = -\frac{\int_{-\infty}^{\infty}(y-x)\Theta(y,0)G(x,y)dy}{t\int_{-\infty}^{\infty}\Theta(y,0)G(x,y)dy}$$

The logical computation would be:

1. Compute $\Theta(y,0)$ using $u(y,0)$
2. Compute $u(x,t)$ using $\Theta(y,0)$

However, the magnitude of $\Theta(y,0)$ may become huge or vanishingly small, usual programming languages cannot deal with it. That is why the Cole-Hopf transformation may not seem really useful.

Numerical schemes

We can express the constant α in $\Theta(y,0)$ depending on x. Taking $\alpha = x$, we can rewrite $\Theta(y,0)$ as

$$\Theta(y,0;x) := \Theta(y,0) = e^{-\frac{1}{2\nu}\int_x^y u(\xi,0)d\xi}$$

It satisfies:

- $\Theta(x,0;x) = 1$
- $|u(y,0)| \leq C_1 \implies \Theta(y,0;x) \leq e^{C_2|y-x|}$

On the other hand $G(x,y)$ decays double-exponentially as $|y-x|$ increases.

Therefore, ΘG and $(y-x)\Theta G$ are effectively zero for $|y-x| \gg 1$ and, if $\nu t \ll 1$, we can compute $u(x,t)$ using $\Theta(y,0;x)$ in the neighborhood of $y = x$.

Then, we will just use $u(x,t)$ as the initial data for the following step.

Numerical schemes

Now, we just need approximate the initial data $u(\xi, 0)$ with respect to space.

We will consider a uniform discretization $x_k = k\Delta x$ and a sufficiently small time-step Δt that satisfies $\frac{(\Delta x)^2}{4\nu\Delta t} \gg 1$, so that only the data of $\Theta(y, 0; x_k)$ in $[x_{k-1}, x_{k+1}]$ is necessary for the computation of $u(x_k, \Delta t)$.

Some additional notation:

- $\eta = \xi - x_k$
- $u_k = u(x_k, 0)$

We shall consider three different approaches.

- Using piecewise linear polynomials ($u(x_k + \eta, 0) = b\eta + a$)
- Using cubic polynomials ($u(x_k + \eta, 0) = c\nu^3 + b\eta + a$)
- Using quartic polynomials ($u(x_k + \eta, 0) = e\nu^4 + d\nu^3 + c\nu^2 + b\nu + a$)

Numerical schemes

Scheme A:

Piecewise linear polynomials

$$u(x_k+\eta,0)=\pm\frac{u_{k\pm1}-u_k}{\Delta x}\eta+u_k,\ \ 0<\pm\eta<\Delta x$$

Therefore, the exponent of ΘG becomes a piecewise quadratic polynomial of ν and, if t satisfies that $1+b_{\pm}\Delta t>1$, the coefficient of η^2 is negative and the integration in the formula for $u(x,\Delta t)$ can be done analytically. Even the integration on $(-\infty,\infty)$ can be done safely.

Numerical schemes

Scheme B:

Piecewise cubic polynomials

$$u(x_k+\eta,0)=\frac{u_{k+1}-2u_k+u_{k-1}}{(\Delta x)^2}\eta^3+\frac{u_{k+1}-u_{k-1}}{2\Delta x}+u_k,\ \ 0<\pm\eta<\Delta x$$

For this case, no general analytical formula can be obtained, as

$$\Theta(x_k+\nu,0;x_k)G(x_k,x_k+\eta)=e^{-\frac{c\eta^3}{6\nu}}e^{-\frac{1}{4\nu\Delta t}[(1+b\Delta t)\eta^2+2a\Delta t\nu]}$$

For small Δt, we can use the approximation $e^{-\frac{c\eta^3}{6\nu}}\approx 1-c\eta^3/6\nu$, so that the integration for $u(x,\Delta t)$ can already be computed analytically.

Numerical schemes

So, we have:

$$u(x_k, \Delta t) = u(x_k, 0) + \frac{s_1 \Delta t + s_2 \Delta t^2 + s_3 \Delta t^3 + s_4 \Delta t^4}{k_0 + k_1 \Delta t + k_2 \Delta t^2 + k_3 \Delta t^3 + k_4 \Delta t^4}$$

with

$$k_0 = 6\nu, \;\; k_1 = 24b\nu, \;\; k_2 = (36b^2 + 6ac)\nu,$$
$$k_3 = a^3c + (24b^3 + 12abc)\nu, \;\; k_4 = a^3bc + 6(b^4 + ab^2c)\nu,$$
$$s1 = -6ab\nu + 12c\nu^2, \;\; s_2 = (-18ab^2 + 6a^2c)\nu + 24bc\nu^2,$$
$$s3 = -18ab^3\nu + 12b^2c\nu^2, \;\; -a^4bc - 6(ab^4 + a^2b^2c)\nu$$

Numerical schemes

Scheme C:

An extension of the scheme B, using piecewise quartic polynomial approximation by a five point formula, that results on:

$$u(x_k, \Delta t) = u(x_k, 0) + \frac{p_0 + p_1\Delta t + p_2\Delta t^2 + p_3\Delta t^3 + p_4\Delta t^4 + p_5\Delta t^5 + p_6\Delta t^6}{q_0 + q_1\Delta t + q_2\Delta t^2 + q_3\Delta t^3 + q_4\Delta t^4 + q_5\Delta t^5 + q_6\Delta t^6}$$

Outline

Some results

We will make two numerical tests:

- P1: $u(x,0) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$
- P2: $u(x,0) = 1 - \sin(\pi x)$

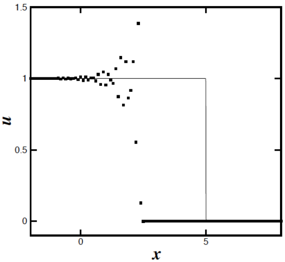
Some results

L_1 error at $t = 1$ versus Δx in P1 for $\nu = 0.1$ and $\Delta t = \Delta x^2$.

Δx	Scheme-A	Scheme-B	Scheme-C
0.1	3.84×10^{-2}	2.64×10^{-3}	2.23×10^{-4}
0.05	3.88×10^{-2}	6.45×10^{-4}	1.37×10^{-5}
0.025	3.89×10^{-2}	1.60×10^{-4}	8.52×10^{-7}
0.0125	3.89×10^{-2}	4.00×10^{-5}	5.32×10^{-8}

Scheme B and Scheme C are nearly second and four order approximations, whereas Scheme A error does not seem to converge.

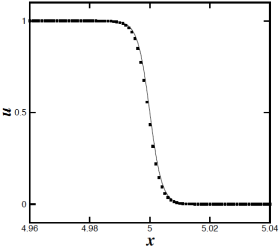
Some results



Scheme C

$\nu = 10^{-3}$, $t = 10$,

$\Delta x = 0.1$, $\Delta x = 0.01$



Scheme C

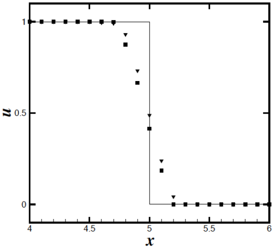
$\nu = 10^{-3}$, $t = 10$,

$\Delta x = 10^{-3}$, $\Delta t = 10^{-4}$

Some results

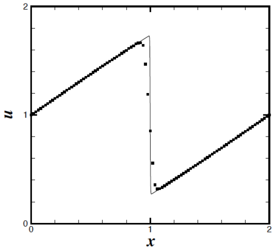
Modification for Scheme B and Scheme C.

We increase ν to $C_1\Delta x$ locally in the computation of $u(x_k, t)$ if the corresponding slopes $b_\pm = \pm\frac{u_{k\pm1}-u_k}{\Delta x}$ satisfy $b_+b_- < 0$ or $|b_\pm| > C_2\Delta x$, for $C_1, C_2 \in \mathbb{R}^+$.



$\nu = 10^{-5},\ t = 10,\ \Delta x = 0.1,$
$\Delta t = 0.01,\ C_1 = 1,\ C_2 = 1.5$

Some results



$\nu = 10^{-3},\ t = 1,\ \Delta x = 0.02,$
$\Delta t = 0.002,\ C_1 = 1,\ C_2 = 1.5$

Bibliography

K. Sakai and I. Kimura, *A numerical scheme based on a solution of nonlinear advection-diffusion equations*, Journal of Computational and Applied Mathematics, vol. 173, 2005, pp. 39-55.

Thanks for your attention!