# Dispersion for the Schrodinger Equation on discrete trees

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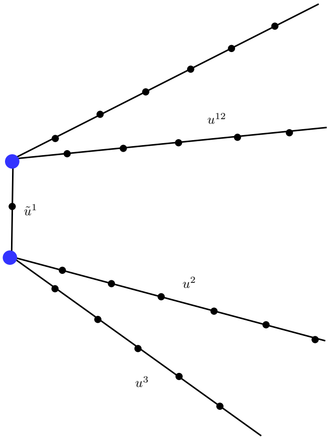
April 11 2012

# Preliminaries

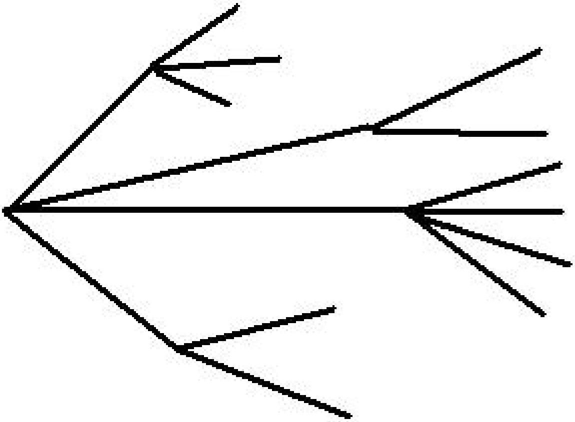
Let $\Gamma = (V, E)$ be a graph, denoting $V$ the set of vertices and $E$ the set of edges, assumed to be segments which are either finite or infinite in one direction only.
Each edge $\alpha \in E$ is parametrized by $I_\alpha$, which is $\{1, 2, \ldots N_\alpha\}$ or $\{1, 2, \ldots\}$.
Each vertex $v \in V$ is the initial point for $m_v$ edges:
$\alpha_i^v$ $(1 \leq i \leq m_v)$ and (except for the origin) the final point for a single edge, $\alpha_0^v$.


$u^{12}$
$\tilde{u}^1$
$u^2$
$u^3$

We indentify the funtions defined on the graph by the families $(u^\alpha)_{\alpha\in E}$ such that $u^\alpha : I_\alpha \to \mathbb{C}$.
If the $v$ vertex is: the final point of the $\alpha$ edge and the origin of the $\alpha_1,\dots,\alpha_m$ vertices, then we extend these functions by setting

$$u^\alpha(N_\alpha+1) = u^{\alpha_i}(0) = \frac{u^\alpha(N_\alpha) + u^{\alpha_1}(1) + \dots u^{\alpha_m}(1)}{m+1}, \qquad \forall\, 1\le i\le m \tag{1}$$

By chosing an origin (the tree becomes oriented) the formula is without the $\alpha$ term, and divide by $m$ instead.



The motivation for studying thin structures comes from mesoscopic physics and nanotechnology. Mesoscopic systems are those that have some dimensions which are too small to be treated using classical physics while they are too large to be considered on the quantum level only. The quantum wires are physical systems with two dimensions reduced to a few nanometers.

The continuous case has been investigated before by Liviu Ignat and Valeria Banica, while in the discrete setting: L.I. and Diana Stan coupled two discrete equations even allowing different coefficients.

We consider the LSE equation on Γ:

$$\begin{cases} i\partial_t u^\alpha(t,j) + \Delta u^\alpha(t,j) = 0, & t \neq 0 \\ u^\alpha(0,j) = \varphi^\alpha(j), & \alpha \in E, \quad j \in I_\alpha. \end{cases} \tag{2}$$

The functions $u^\alpha$ obey the "continuity" and coupling conditions defined in (1).
We define the laplacian $\Delta$ by

$$\Delta u_j^\alpha = u_{j+1}^\alpha - 2u_j^\alpha + u_{j-1}^\alpha.$$

## Theorem 1.1

*For every* $\varphi \in l^2$*, the system* (10) *has a unique solution* $u = (u^\alpha)_{\alpha \in E} \in C(\mathbb{R}, l^2)$*. Additionally, there exists* $C > 0$ *for which*

$$||u(t)||_{l^\infty} \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} \qquad \forall\ \varphi \in l^1. \tag{3}$$

From this, we get the following consequence:

## Theorem 1.2

*For every* $\varphi \in l^2$*, the solution* $u$ *satisfies*

$$||u||_{L^q(\mathbb{R}, l^r)} \leq C_{q,r} ||\varphi||_{l^2} \tag{4}$$

*for the* $(q, r)$ *pairs for which* $1/q \leq 1/3(1/2 - r)$*.*

# The spectrum

## Proposition 1.3

*The spectrum of the discrete laplacian on* $\Gamma$ *is* $\sigma(\Delta) = [-4, 0]$.

Consider the resolvent $R_\lambda = (\Delta - \lambda)^{-1} \in B(l^2)$ for $\lambda \in \mathbb{C} \setminus [-4, 0]$.

## Proposition 1.4

*For* $f \in l^2$, *the resolvent* $R_\lambda f = u$ *is given by*

$$\begin{cases} u_n^\alpha = u_0^\alpha \cdot r_1^n + S_n^\alpha, & ,\alpha = \text{ infinite edge },\ n \in I_\alpha \\ u_n^\alpha = a^\alpha \cdot r_1^n + b^\alpha \cdot r_2^n + S_n^\alpha, & ,\alpha = \text{ finite edge},\ n \in I_\alpha. \end{cases} \tag{5}$$

*where* $S_n^\alpha$ *is defined by*

$$S_j^\alpha = \frac{1}{r_1 - \frac{1}{r_1}} \sum_{k \in I_\alpha} \left[ r_1^{|j-k|} - r_1^{j+k} \right] f_k^\alpha. \tag{6}$$

Now, $r_1, r_2$ are the solutions of the equation $r^2 - (2+\lambda)r + 1 = 0$ for $\lambda = x - i\varepsilon$:

## Proposition 1.5

- *Let* $x \in [-4, 0]$ *and* $\varepsilon > 0$. *For any* $f \in l^2$ *and any* $\alpha \in E$:

$$[R_{x-i\varepsilon} - R_{x+i\varepsilon}]\, f_n^\alpha = \frac{1}{2i}\,\mathrm{Im}[a^\alpha r_1^n + b^\alpha r_2^n] +$$

$$+ \frac{1}{2i}\sum_{k\in I_\alpha} f_k^\alpha\, \mathrm{Im}\left[\frac{1}{r_1 - \frac{1}{r_1}}\left(r_1^{|n-k|} - r_1^{n+k}\right)\right], \quad n \in I_\alpha \qquad (7)$$

- *For any* $\varphi \in l^1$ *and* $a, b \in \mathbb{R}$ *we have*

$$\lim_{\varepsilon\to 0+}\int_a^b [R_{x-i\varepsilon} - R_{x+i\varepsilon}]\,\varphi_n\,\mathrm{d}x = \int_a^b [R_{x-} - R_{x+}]\,\varphi_n\,\mathrm{d}x, \quad n \in I_\alpha \qquad (8)$$

The last term in the RHS is given by

$$2i\,[R_{x-}-R_{x+}]\,\varphi_n^\alpha = \mathrm{Im}[a_-^\alpha r^n + b_-^\alpha \overline{r}^n] + \sum_{k\in I_\alpha} \varphi_k^\alpha \,\mathrm{Im}\,\frac{1}{r-\frac{1}{r}}\left(r^{|n-k|}-r^{n+k}\right) \tag{9}$$

such that $r$ is the solution with $\mathrm{Im}\, r \geq 0$ of the equation
$r^2-(2+x)r+1=0$
This shows that $R_{x-}-R_{x+}$ is well-defined as an operator in $B(l^1, l^\infty)$.

We return to the equation

$$\begin{cases} i\partial_t u^\alpha(t,j) + \Delta u^\alpha(t,j) = 0, & t \neq 0 \\ u^\alpha(0,j) = \varphi(j), & \alpha \in E, \quad j \in I_\alpha. \end{cases} \tag{10}$$

## Theorem 1.6

*For any $\varphi \in l^2$, there exists a unique solution $u = (u^\alpha)_{\alpha \in E} \in C(\mathbb{R}, l^2)$ of the system (10) which satisfies $||u(t)||_{l^2} = ||\varphi||_{l^2}$. Moreover, if $\varphi \in l^1$ then $u$ can be represented as*

$$u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \frac{1}{2\pi i}\int_I e^{it\lambda}\left[R_{\lambda-} - R_{\lambda+}\right]\varphi_n \,\mathrm{d}\lambda. \tag{11}$$

*Proof.* We use Stone's (or Hille-Yosida) theorem to show that there exists a unique solution $u \in C(\mathbb{R}, l^2)$ of equation (10), given by

$$u(t) = e^{it\Delta}\varphi, \tag{12}$$

where the operators $e^{it\Delta}$ are defined by power series or by functional calculus. They are unitary operators, so

$$||u(t)||_{l^2} = ||\varphi||_{l^2}.$$

$$u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \int_{\sigma(\Delta)} e^{it\lambda}\, \mathrm{d}\mu_{\varphi,n}, \tag{13}$$

$$\mu_{\varphi,n}(a,b) = \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} R_{(x-i\varepsilon)}T - R_{(x+i\varepsilon)}T\, \mathrm{d}x$$

So, we get

$$u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \frac{1}{2\pi i}\int_I e^{it\lambda}\left[R_{\lambda-} - R_{\lambda+}\right]\varphi_n\, \mathrm{d}\lambda. \quad \square$$

# The key lemma

## Lemma 1.7

*For any* $\varphi \in l^1$ *and any* $t \in \mathbb{R}$, $n \in \mathbb{Z}$ *the following inequalities are true:*

$$\left| \int_I e^{it\lambda} u^{\alpha}_{0,-} r^n \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} \tag{14}$$

$$\left| \int_I e^{it\lambda} a^{\alpha}_{-} r^n \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} \tag{15}$$

$$\left| \int_I e^{it\lambda} b^{\alpha}_{-} \overline{r}^n \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} \tag{16}$$

We will discuss this lemma after the main result.

# Proof of the main result

We use (9) and (11) and observe that, in order to get the inequality (3), it suffices to check that for all $n$ and $k$

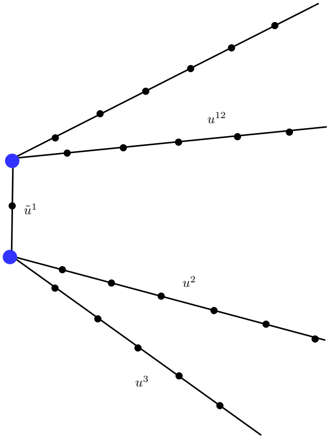
$$\left|\int_I e^{it\lambda}\,\mathrm{Im}\, u^{\alpha}_{0,-} r^n \,\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt[3]{1+|t|}}\,||\varphi||_{l^1} \qquad (17)$$

$$\left|\int_I e^{it\lambda}\,\mathrm{Im}\, a^{\alpha}_{-} r^n \,\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt[3]{1+|t|}}\,||\varphi||_{l^1}\,, \quad \left|\int_I e^{it\lambda}\,\mathrm{Im}\, b^{\alpha}_{-} \overline{r}^n \,\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt[3]{1+|t|}}\,| \qquad (18)$$

$$\left|\int_I e^{it\lambda}\,\mathrm{Im}\left[\frac{1}{r-\frac{1}{r}}\left(r^{|n-k|}-r^{n+k}\right)\right]\mathrm{d}\lambda\right| \leq \frac{C}{\sqrt[3]{1+|t|}}. \qquad (19)$$

Inequality 19 was proved in [3], (4.13) , in a more general case by using Van der Corput's lemma.
It is sufficient to prove 17 and 18 without the imaginary part Im because, by chosing $-t$ instead of $t$ and conjugating both, we get the inequalities with additional Im. So, by using the lemma stated before, the result follows. □

# A particular case



To prove the inequalities in the lemma for this particular case, we have to determine $u_0^{11}, u_0^{12}, u_0^2, u_0^3, a = a^1$ *and* $b = b^1$ from the system

$$\begin{cases} u_0^2 - u_0^3 = 0 \\ u_0^2 - a - b = 0 \\ (r-1)u_0^2 + (r-1)u_0^3 + (r-1)a + (r_2-1)b = -S_1^1 - S_1^2 - S_1^3 = K_3 \\ r^{N+1}a + r_2^{N+1}b - u_0^{11} = 0 \\ u_0^{11} - u_0^{12} = 0 \\ r^N a + r_2^N b + \left(r - \frac{3}{2}\right) u_0^{11} + \left(r - \frac{3}{2}\right) u_0^{12} = -\left(S_N^1 + S_1^{11} + S_1^{12}\right) = K_6. \end{cases} \tag{20}$$

We have the following matrix $M$ coresponding to the variables: $x = [u_0^2, u_0^3, a, b, u_0^{11}, u_0^{12}] = [x_i]$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ r_1 - 1 & r_1 - 1 & r_1 - 1 & r_2 - 1 & 0 & 0 \\ --- & --- & --- & & & \\ 0 & 0 & r_1^{N+1} & r_2^{N+1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & r_1^N & r_2^N & r_1 - \frac{3}{2} & r_1 - \frac{3}{2} \end{array}\right] \quad K = \begin{pmatrix} 0 \\ 0 \\ K_3 \\ 0 \\ 0 \\ K_6 \end{pmatrix}$$

We see that

$$\det M = (r_1 - 1)^2 \cdot r_2^{N+1} \left[9 - r_1^{2N}(2r_1 - 1)^2\right] = (r_1 - 1)^2 \cdot (r_1 + 1) h(r_1),$$

for a polynomial function $g(x) = x^{N+1} h(x)$ which does not vanish on $\mathbb{T}$.

To determine the elements $x_i$ we have to insert the column $K$ in the place of column $i$ in $M$ and to compute this determinant, denoted $D_i(r_1)$, obtaining $x_i = \frac{D_i}{\det M}$.

## Lemma 1.8

*For every* $1 \leq i \leq 6$ *and every* $\alpha \in E$ *there is a polynomial* $q_1^{\alpha,i}(x,y)$ *, and another polynomial* $q_{N_\alpha}^{\alpha,i}(x,y)$ *when* $\alpha$ *is finite such that*

$$D_i(r_1) = (r_1 - 1)\left[\sum_{\alpha \in E} S_1^\alpha \cdot q_1^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) + \sum_{\alpha - finite} S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}\left(r_1, \frac{1}{r_1}\right)\right].$$

*Proof.* We have $r_2 = \frac{1}{r_1}$, so when $r_1 = 1$ also $r_2 = 1$ follows. We must check that $D_i(r_1) = 0$ for $r_1 = 1$:

- If $K$ enters in a different column than 3 or 4, then the columns 3 and 4 are equal;
- If $K$ enters in column 3 or 4, we easily make a linear combination of columns that is 0. □

We get an expression for $x_i \in \{u_0^2, u_0^3, a, b, u_0^{11}, u_0^{12}\}$:

$$x_i = \sum_{\alpha \in E} \frac{S_1^\alpha \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} + \sum_{\alpha - finite} \frac{S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)}. \tag{21}$$

We write (6) for these cases, and we will plug (22) and (23) into (21).

$$S_1^\alpha = -\sum_{k \in I_\alpha} r_1^k \varphi_k^\alpha \tag{22}$$

$$S_{N_\alpha}^\alpha = -r_1^{N_\alpha} \sum_{k \in I_\alpha} \frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \varphi_k^\alpha, \qquad (if\ \ \alpha - finite). \tag{23}$$

## Lemma 1.9

*There are constants $C$ such that for every $m \in \mathbb{Z}$, $\alpha \in E$, $k \in I_\alpha$ and $t \in \mathbb{R}$, the following inequalities are true:*

$$i) \quad \left| \int_I e^{it\lambda} \frac{r_1^m}{(r_1-1)(r_1+1)g(r_1)} \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}}$$

$$ii) \quad \left| \int_I e^{it\lambda} \frac{q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^m \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}}$$

$$iii) \left| \int_I e^{it\lambda} \frac{\frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^m \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \quad \textit{if } \alpha\text{−finite edge.}$$

*Proof of i.* The function $g$ is a polynomial which does not vanish on the unit circle so, using Wiener's theorem (18.21 from [4]), $1/g$ has an absolutely and uniformly convergent Fourier series

$$\frac{1}{g(r_1)} = \sum_{p\geq 0} a_p r_1^p, \qquad \sum_{p\geq 0} |a_p| < A.$$

This reduces the proof to

$$\left| \int_I e^{it\lambda} \frac{r_1^m}{(r_1-1)(r_1+1)} \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}}. \qquad (m \in \mathbb{Z}).$$

We now write $r_1 = e^{i\theta}$ and hence

$$\lambda = 2(\cos\theta + 1) \quad \textit{and} \quad (r_1-1)(r_1+1) = i\sin\theta e^{i\theta}.$$

By making a change of variable such that we integrate over $\theta \in [0, \pi]$, the last integral becomes

$$\left| \int_0^{\pi} e^{2it(\cos\theta - 1)} e^{im\theta} \, \mathrm{d}\theta \right| \leq \frac{C}{\sqrt[3]{1 + |t|}} \qquad (m \in \mathbb{Z}).$$

This last estimate was proved at (4.14) in [3], by using Van der Corput's lemma. $\square$

# Proof of Lemma 1.7

For $x_i \in \{u_0^2, u_0^3, a, b, u_0^{11}, u_0^{12}\}$ we have to prove that

$$\left|\int_I e^{it\lambda} x_i r_1^n \, \mathrm{d}\lambda\right| \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} \qquad (n \in \mathbb{Z}).$$

We plug (22) and (23) in (21) and by using Lemma 1.9, we get

$$\left|\int_I e^{it\lambda} x_i r_1^n \, \mathrm{d}\lambda\right| \leq \sum_{\alpha \in E} \sum_{k \in I_\alpha} |\varphi_k^\alpha| \left|\int_I e^{it\lambda} \frac{r_1^k \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^n \, \mathrm{d}\lambda\right| +$$

$$+ \sum_{\alpha - finite} \sum_{k \in I_\alpha} |\varphi_k^\alpha| \left|\int_I e^{it\lambda} \frac{r_1^{N_\alpha} \frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^n \, \mathrm{d}\lambda\right| \leq$$

$$\leq \frac{C}{\sqrt[3]{1+|t|}} \left[\sum_{\alpha \in E} \sum_{k \in I_\alpha} |\varphi_k^\alpha| + \sum_{\alpha - finite} \sum_{k \in I_\alpha} |\varphi_k^\alpha|\right] \leq \frac{C}{\sqrt[3]{1+|t|}} ||\varphi||_{l^1} . \square$$

# The general case

Like in the particular case, we have to determine the coefficients from Lemma 1.7.
By denoting the determinant of the system of graph $\Gamma_n$ by $D_n$ we get the following recurrence relation:

$$D_n = D_{n-1} \cdot r_2^{N+1} \cdot (m+1) \cdot (r_1 - 1) \cdot \left[ 1 - \left( \frac{r_1}{r_2} \right)^{N+1} \frac{m - r_2}{m+1} \frac{\tilde{D}_{n-1}}{D_{n-1}} \right], \quad (24)$$

where $\tilde{D}_n$ is another similar determinant.

**Lemma 2**

*We have*

$$D_n = (r_1 - 1)^n \cdot (r_1 + 1)^{n-1} \cdot h(r_1) \tag{25}$$

*for $h$ a rational function which does not vanish on $\mathbb{T}$.*

## Lemma 3

*We have*

$$D_n^i = (r_1 - 1)^{n-1} \cdot (r_1 + 1)^{n-2} \cdot q_i^n(r_1, r_2) \tag{26}$$

*for*

$$q_i^n(x,y) = \sum_\alpha S_1^\alpha \cdot q_1^{\alpha,i}(x,y) + S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(x,y)$$

*where $q_1^{\alpha,i}$ si $q_{N_\alpha}^{\alpha,i}$ are polynomials.*

From here on, the solution is identical with the previous particular case, since we have the expansion for $\frac{1}{h}$ and the factors $D_n$ and $D_n^i$ in $x_i$ cancel, which leads us to the same integrals as before.

# THANKS for your attention !!!

# References


1. W. Rudin, *Functional analysis*, McGraw-Hill Science, 1991.
2. N. Dunford and J.T. Schwartz, Linear Operators II: Spectral Theory, Interscience, 1963.
3. L.I. Ignat, D. Stan, Dispersive properties for discrete Schrodinger equations, J. Fourier Analysis and Applications
4. W. Rudin , *Real and complex analysis*