Drag of a thin wing and optimal shape to minimize it

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Outline

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Statement of the problem

A thin wing is placed at a small angle of incidence $\alpha$ in a steady supersonic stream, so that the wing is given by:

$$y = \varepsilon f_{\pm}(x) - \alpha x, \quad \text{for } 0 < x < l$$

where $l$ is the length of the wing and $\alpha = \mathcal{O}(\varepsilon)$. 
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Our aim is to:

1. Show that the drag can be approximated by

$$D = \frac{\rho_0 U^2}{B} \int_0^l \left[ (\varepsilon f_\pm'(x) - \alpha)^2 + (\varepsilon f_-'(x) - \alpha)^2 \right] dx$$

2. Confirm that the drag on a flat plate of length $l$ at small angle of incidence $\alpha$ is $\left(2\rho_0 U^2 / B\right)\alpha^2 l$.

3. Obtain the optimal shape for a kite-form cross-section wing with given length and thickness.
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Our aim is to:

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   \[
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2. Confirm that the drag on a flat plate of length \( l \) at small angle of incidence \( \alpha \) is \( (2\rho_0 U^2 / B)\alpha^2 l \).

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We will consider that the wing is placed in an inviscid compressible flow of a perfect gas. Therefore, we will start from the following equations:

- **Continuity equation:**
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{Cont.}
  \]

- **Euler’s equation:**
  \[
  \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p \tag{Euler}
  \]

- **Energy equation:**
  \[
  \rho \frac{de}{dt} = \rho \frac{d\rho}{dt} + \nabla \cdot (k \nabla T) + \rho \frac{dQ}{dt} \tag{Energy}
  \]

- **Perfect gas law:**
  \[
  p = \rho RT \tag{Gas}
  \]

Besides the necessary initial state, we could also have some boundary condition (e.g., thing wing):

\[
 f(x, t) = 0 \implies \frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0
\]
Let us take pressure, density and temperature as perturbations:

\[ p = p_0 + \bar{p} \quad \rho = \rho_0 + \bar{\rho} \quad T = T_0 + \bar{T} \]

Assuming that the barred quantities are small and neglecting *squares*:

(Cont.) \[ \frac{\partial \bar{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} + \nabla \cdot (\bar{\rho} \mathbf{u}) = 0 \]

(Euler) \[ (\rho_0 + \bar{\rho}) \frac{\partial \mathbf{u}}{\partial t} + (\rho_0 + \bar{\rho})(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \bar{p} = 0 \]

Suppose gas is in a state of uniform motion along the \( x \) axis with some small disturbance, i.e. \( \mathbf{u} = U\mathbf{i} + \bar{\mathbf{u}} \). Then:

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \bar{\rho} + \rho_0 \nabla \cdot \bar{\mathbf{u}} = 0 \quad \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \bar{\mathbf{u}} + \frac{1}{\rho_0} \nabla \bar{p} = 0 \]

Note that \( c_p - c_v = R \) and denote \( \gamma = \frac{c_p}{c_v} \). Assuming also that \( k \approx 0 \) (no heat conduction):

\[ (\text{Energy}) + (\text{Gas}) \Rightarrow \frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \Rightarrow \bar{p} = \frac{\gamma p_0}{\rho_0} \bar{\rho} \]

\[ c_0^2 \]
Therefore, using the three simplified equations:

$$\nabla^2 \varphi = \frac{1}{c_0^2} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 \varphi,$$

for $\varphi = \bar{p}, \bar{\rho}, \bar{u}$

In particular, for steady flows we have:

$$\nabla^2 \varphi = \frac{U^2}{c_0^2} \frac{\partial^2 \varphi}{\partial x^2} \quad \text{(Wave)}$$

We denote $M = U/c_0$, which is called Mach number. Depending on the value of $M$, we distinguish two main cases:

- If $M > 1$, the equation is hyperbolic and the flow is said to be supersonic.
- If $M < 1$, the equation is elliptic and the flow is said to be subsonic.
Compressible flow past a thin wing

Now, let us consider the perturbation of $u$ of the type $\bar{u} = \varepsilon \nabla \phi$. Then, $\phi$ also satisfies (Wave) and from (Euler) we have:

$$p = p_0 - \varepsilon \rho_0 U \frac{\partial \phi}{\partial x}$$

Let us apply the flow to a two-dimensional thin wing, which is nearly aligned with the flow. The wing is given by:

$$y = \varepsilon f_{\pm}(x) - \alpha x, \quad \text{for } 0 < x < l$$

Therefore, the boundary condition is:

$$f'_{\pm}(x) = \frac{\partial \phi/\partial y}{U + \varepsilon \frac{\partial \phi}{\partial x}} + \frac{\alpha}{\varepsilon}, \quad \text{on } y = \varepsilon f_{\pm}(x) - \alpha x$$

And, if we suppose $\alpha = O(\varepsilon)$ (i.e. $|\alpha| \leq C\varepsilon$), we have the linear approximation:

$$Uf'_{\pm}(x) = \frac{\partial \phi}{\partial y} + UC, \quad \text{on } y = 0_{\pm}, \quad \text{for } 0 < x < l$$

(BC)
Supersonic case \((M > 1)\)

When \(M > 1\), the solution to

\[
M^2 \frac{\partial^2 \phi}{\partial x} = \nabla^2 \phi
\]

is of the form:

\[
\phi = F(x - By) + G(x + By)
\]

where \(B^2 = M^2 - 1\). Assuming that there will be no upstream influence due to the wing, we impose:

\[
\frac{\partial \phi}{\partial x} \bigg|_{x=0} = \phi \bigg|_{x=0} = 0
\]

Therefore:

\[
\begin{cases} 
  \phi = \phi_+ = F(x - By), & y > 0 \\
  \phi = \phi_- = G(x + By), & y < 0
\end{cases}
\]

and applying boundary condition (BC) on the wing:

\[
\begin{cases} 
  -BF'(x) + UC = Uf'_+(x) \quad , \text{for } 0 < x < l \\
  BG'(x) + UC = Uf'_-(x)
\end{cases}
\]
Supersonic case $(M > 1)$

So the solution to that is:

$$
\phi_+ = -\frac{U}{B} f_+(x - By) + \frac{UC}{B} x, \quad \text{for } 0 < x - By < l, y > 0
$$

$$
\phi_- = \frac{U}{B} f_-(x + By) - \frac{UC}{B} x, \quad \text{for } 0 < x - By < l, y < 0
$$

So now we can easily compute the drag:

$$
D = \int_0^l \left( p_+ \frac{dy_+}{dx} - p_- \frac{dy_-}{dx} \right) dx
$$

$$
= \int_0^l \left[ \left( p_0 - \varepsilon \rho_0 U \frac{\partial \phi_+}{\partial x} \right) (\varepsilon f_+(x) - \alpha) - \left( p_0 - \varepsilon \rho_0 U \frac{\partial \phi_-}{\partial x} \right) (\varepsilon f_-(x) - \alpha) \right] dx
$$

$$
= \varepsilon p_0 \int_0^l (f_+(x) - f_-(x)) \, dx + \varepsilon^2 \rho_0 U \int_0^l \left( - \frac{\partial \phi_+}{\partial x} f_+(x) + \frac{\partial \phi_-}{\partial x} f_-(x) \right) \, dx
$$

$$
- \alpha \varepsilon \rho_0 U \int_0^l \left( - \frac{\partial \phi_+}{\partial x} + \frac{\partial \phi_-}{\partial x} \right) \, dx
$$
Drag of the wing

Recovering the expressions for $\phi$, we finally obtain:

$$D = \varepsilon^2 \rho_0 U \int_0^l \left[ \left( \frac{U}{B} f'_+ (x) - \frac{UC}{B} \right) f'_+ (x) + \left( \frac{U}{B} f'_- (x) - \frac{UC}{B} \right) f'_- (x) \right] \, dx$$

$$- \alpha \varepsilon \rho_0 U \int_0^l \left[ \left( \frac{U}{B} f'_+ (x) - \frac{UC}{B} \right) + \left( \frac{U}{B} f'_- (x) - \frac{UC}{B} \right) \right] \, dx$$

$$= \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[ (f'_+ (x))^2 + (f'_- (x))^2 \right] \, dx$$

$$- \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[ C f'_+ (x) + C f'_- (x) \right] \, dx$$

$$- \frac{\alpha \varepsilon \rho_0 U^2}{B} \int_0^l \left[ (f'_+ (x) - C) + (f'_- (x) - C) \right] \, dx$$

$$= \frac{\rho_0 U^2}{B} \int_0^l \left[ (\varepsilon f'_+ (x) - \alpha)^2 + (\varepsilon f'_- (x) - \alpha)^2 \right] \, dx$$

Therefore, if we take $\varepsilon \to 0$, the drag on a flat plate of length $l$ at a small angle of incidence $\alpha$ is $(2 \rho_0 U^2 / B) \alpha^2 l$. 

Example

Suppose now that $\alpha = 0$ and, for a given thickness $h$, let us consider a wing with a cross section given by:

$$f_+ = -f_- = \begin{cases} mx, & 0 < x < \frac{h}{m} \\ \frac{h(l-x)}{l-h/m}, & \frac{h}{m} < x < l \end{cases}$$

Using the formula obtained before, we have that the drag of this wing is:

$$D = \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[ \left(f_+(x)\right)^2 + \left(f_-(x)\right)^2 \right] \, dx$$

$$= \frac{2\varepsilon^2 \rho_0 U^2}{B} \left[ \int_0^{\frac{h}{m}} m^2 \, dx + \int_{\frac{h}{m}}^l \left(\frac{h}{l-h/m}\right)^2 \, dx \right]$$

$$= \frac{2\varepsilon^2 \rho_0 U^2}{B} \frac{m^2 hl}{ml - h}$$
Minimum drag

For $l$ and $h$ fixed, the minimum drag is achieved for:

$$\frac{\partial}{\partial m} \left( \frac{2 \varepsilon^2 \rho_0 U^2}{B} \frac{m^2 hl}{ml - h} \right) = 0 \implies \frac{h}{m} = \frac{l}{2}$$

that corresponds to a diamond shaped wing.
Thanks for your attention!

Zorionak eta urte berri on!!