Drag of a thin wing and optimal shape to minimize it

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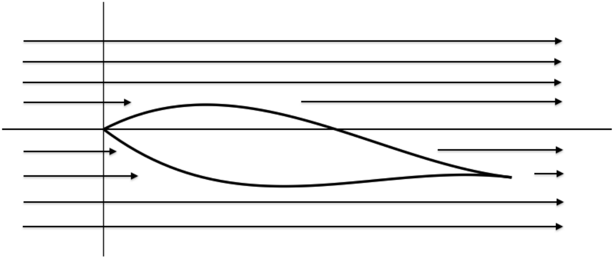
Outline

Statement of the problem

A thin wing is placed at a small angle of incidence α in a steady supersonic stream, so that the wing is given by:

$$y = \varepsilon f_{\pm}(x) - \alpha x, \qquad \text{for } 0 < x < l$$

where l is the length of the wing and $\alpha = \mathcal{O}(\varepsilon)$.



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Our aim is to:

1. Show that the drag can be approximated by

$$D = \frac{\rho_0 U^2}{B} \int_0^l \left[\left(\varepsilon f'_+(x) - \alpha\right)^2 + \left(\varepsilon f'_-(x) - \alpha\right)^2 \right] dx$$

2. Confirm that the drag on a flat plate of length l at small angle of incidence α is $(2\rho_0 U^2/B)\alpha^2 l$.
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H. Ockendon, J. R. Ockendon, *Waves and Compressible Flow*, Text in Applied Mathematics, vol. 47, Springer, 2004

Equations of the model

We will consider that the wing is placed in an inviscid compressible flow of a perfect gas. Therefore, we will start from the following equations:

- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{(Cont.)}$$

- Euler's equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \qquad \text{(Euler)}$$

- Energy equation:

$$\rho \frac{de}{dt} = \frac{p}{\rho}\frac{d\rho}{dt} + \nabla \cdot (k \nabla T) + \rho \frac{dQ}{dt} \qquad \text{(Energy)}$$

- Perfect gas law:

$$p = \rho R T \qquad \text{(Gas)}$$

Besides the necessary initial state, we could also have some boundary condition (e.g., thing wing):

$$f(\mathbf{x}, t) = 0 \Longrightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0$$

Acoustics

Let us take pressure, density and temperature as perturbations:

$$p = p_0 + \bar{p} \qquad \rho = \rho_0 + \bar{\rho} \qquad T = T_0 + \bar{T}$$

Assuming that the barred quantities are small and neglecting *squares*:

$$\text{(Cont.)} \Rightarrow \frac{\partial \bar{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} + \nabla \cdot (\bar{\rho}\mathbf{u}) = 0$$

$$\text{(Euler)} \Rightarrow (\rho_0 + \bar{\rho})\frac{\partial \mathbf{u}}{\partial t} + (\rho_0 + \bar{\rho})(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \bar{p} = 0$$

Suppose gas is in a state of uniform motion along the x axis with some small disturbance, i.e. $\mathbf{u} = U\mathbf{i} + \bar{\mathbf{u}}$. Then:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\bar{\rho} + \rho_0 \nabla \cdot \bar{\mathbf{u}} = 0 \qquad \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\bar{\mathbf{u}} + \frac{1}{\rho_0}\nabla \bar{p} = 0$$

Note that $c_p - c_v = R$ and denote $\gamma = \frac{c_p}{c_v}$. Assuming also that $k \approx 0$ (no heat conduction):

$$\text{(Energy)} + \text{(Gas)} \Rightarrow \frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \Rightarrow \bar{p} = \underbrace{\frac{\gamma p_0}{\rho_0}}_{c_0^2} \bar{\rho}$$

Acoustics

Therefore, using the three simplified equations:

$$\nabla^2 \varphi = \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \varphi, \quad \text{for } \varphi = \bar{p}, \bar{\rho}, \bar{\mathbf{u}}$$

In particular, for steady flows we have:

$$\nabla^2 \varphi = \frac{U^2}{c_0^2}\frac{\partial^2 \varphi}{\partial x} \qquad \text{(Wave)}$$

We denote $M = U/c_0$, which is called Mach number. Depending on the value of M, we distinguish two main cases:

- If $M > 1$, the equation is hyperbolic and the flow is said to be *supersonic*.
- If $M < 1$, the equation is elliptic and the flow is said to be *subsonic*.

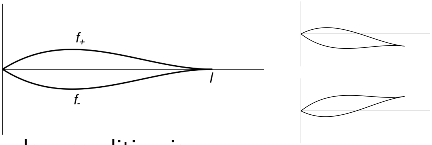
Compressible flow past a thin wing

Now, let us consider the perturbation of $\mathbf{u}$ of the type $\bar{\mathbf{u}} = \varepsilon \nabla \phi$. Then, ϕ also satisfies (Wave) and from (Euler) we have:

$$p = p_0 - \varepsilon \rho_0 U \frac{\partial \phi}{\partial x}$$

Let us apply the flow to a two-dimensional thin wing, which is nearly aligned with the flow. The wing is given by:

$$y = \varepsilon f_{\pm}(x) - \alpha x, \quad \text{for } 0 < x < l$$



Therefore, the boundary condition is:

$$f'_{\pm}(x) = \frac{\partial \phi / \partial y}{U + \varepsilon \; \partial \phi / \partial x} + \frac{\alpha}{\varepsilon}, \quad \text{on } y = \varepsilon f_{\pm}(x) - \alpha x$$

And, if we suppose $\alpha = \mathcal{O}(\varepsilon)$ (i.e. $|\alpha| \leq C\varepsilon$), we have the linear approximation:

$$U f'_{\pm}(x) = \frac{\partial \phi}{\partial y} + UC, \quad \text{on } y = 0_{\pm}, \quad \text{for } 0 < x < l \qquad \text{(BC)}$$

Supersonic case ($M > 1$)

When $M > 1$, the solution to

$$M^2 \frac{\partial^2 \phi}{\partial x} = \nabla^2 \phi$$

is of the form:

$$\phi = F(x - By) + G(x + By)$$

where $B^2 = M^2 - 1$. Assuming that there will be no upstream influence due to the wing, we impose:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \left.\phi\right|_{x=0} = 0$$

Therefore:

$$\begin{cases} \phi = \phi_+ = F(x - By), & y > 0 \\ \phi = \phi_- = G(x + By), & y < 0 \end{cases}$$

and applying boundary condition (BC) on the wing:

$$\begin{cases} -BF'(x) + UC = Uf'_+(x) \\ BG'(x) + UC = Uf'_-(x) \end{cases} \text{, for } 0 < x < l$$

Supersonic case ($M > 1$)

So the solution to that is:

$$\phi_+ = -\frac{U}{B} f_+(x - By) + \frac{UC}{B} x, \quad \text{for } 0 < x - By < l, y > 0$$
$$\phi_- = \frac{U}{B} f_-(x + By) - \frac{UC}{B} x, \quad \text{for } 0 < x - By < l, y < 0$$

So now we can easily compute the drag:

$$\begin{aligned} D &= \int_0^l \left(p_+ \frac{dy_+}{dx} - p_- \frac{dy_-}{dx} \right) dx \\ &= \int_0^l \left[\left(p_0 - \varepsilon \rho_0 U \frac{\partial \phi_+}{\partial x} \right) \left(\varepsilon f_+'(x) - \alpha \right) - \left(p_0 - \varepsilon \rho_0 U \frac{\partial \phi_-}{\partial x} \right) \left(\varepsilon f_-'(x) - \alpha \right) \right] dx \\ &= \varepsilon p_0 \int_0^l \left(f_+'(x) - f_-'(x) \right) dx + \varepsilon^2 \rho_0 U \int_0^l \left(-\frac{\partial \phi_+}{\partial x} f_+'(x) + \frac{\partial \phi_-}{\partial x} f_-'(x) \right) dx \\ &\quad - \alpha \varepsilon \rho_0 U \int_0^l \left(-\frac{\partial \phi_+}{\partial x} + \frac{\partial \phi_-}{\partial x} \right) dx \end{aligned}$$

Drag of the wing

Recovering the expressions for ϕ, we finally obtain:

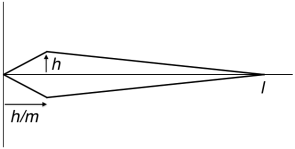
$$\begin{aligned}
D &= \varepsilon^2 \rho_0 U \int_0^l \left[\left(\frac{U}{B} f_+'(x) - \frac{UC}{B} \right) f_+'(x) + \left(\frac{U}{B} f_-'(x) - \frac{UC}{B} \right) f_-'(x) \right] dx \\
&\quad - \alpha \varepsilon \rho_0 U \int_0^l \left[\left(\frac{U}{B} f_+'(x) - \frac{UC}{B} \right) + \left(\frac{U}{B} f_-'(x) - \frac{UC}{B} \right) \right] dx \\
&= \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[\left(f_+'(x) \right)^2 + \left(f_-'(x) \right)^2 \right] dx \\
&\quad - \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[C f_+'(x) + C f_-'(x) \right] dx \\
&\quad - \frac{\alpha \varepsilon \rho_0 U^2}{B} \int_0^l \left[\left(f_+'(x) - C \right) + \left(f_-'(x) - C \right) \right] dx \\
&= \frac{\rho_0 U^2}{B} \int_0^l \left[\left(\varepsilon f_+'(x) - \alpha \right)^2 + \left(\varepsilon f_-'(x) - \alpha \right)^2 \right] dx
\end{aligned}$$

Therefore, if we take $\varepsilon \to 0$, the drag on a flat plate of length l at a small angle of incidence α is $(2\rho_0 U^2/B)\alpha^2 l$.

Example

Suppose now that $\alpha = 0$ and, for a given thickness h, let us consider a wing with a cross section given by:

$$f_+ = -f_- = \begin{cases} mx, & 0 < x < \frac{h}{m} \\ \dfrac{h(l-x)}{l-h/m}, & \frac{h}{m} < x < l \end{cases}$$



Using the formula obtained before, we have that the drag of this wing is:

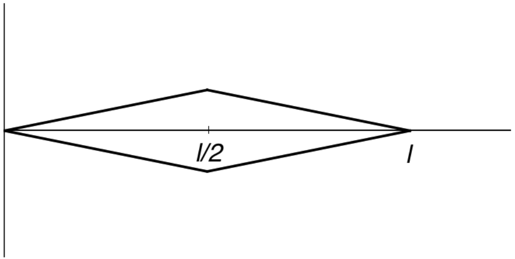
$$\begin{aligned} D &= \frac{\varepsilon^2 \rho_0 U^2}{B} \int_0^l \left[\left(f_+'(x)\right)^2 + \left(f_-'(x)\right)^2 \right] dx \\ &= \frac{2\varepsilon^2 \rho_0 U^2}{B} \left[\int_0^{\frac{h}{m}} m^2 dx + \int_{\frac{h}{m}}^l \left(\frac{h}{l-h/m} \right)^2 dx \right] \\ &= \frac{2\varepsilon^2 \rho_0 U^2}{B} \frac{m^2 hl}{ml-h} \end{aligned}$$

Minimum drag

For l and h fixed, the minimum drag is achieved for:

$$\frac{\partial}{\partial m}\left(\frac{2\varepsilon^2\rho_0 U^2}{B}\frac{m^2 hl}{ml-h}\right) = 0 \Longrightarrow \frac{h}{m} = \frac{l}{2}$$

that corresponds to a diamond shaped wing.



Thanks for your attention!