Analysis of models of non-homogeneous fluids and hyperbolic operators

Francesco Fanelli

SISSA – Sector of Functional Analysis and Applications

UNIVERSITÉ PARIS-EST – Laboratoire d'Analyse et de Mathématiques Appliquées

Basque Center for Applied Mathematics

Bilbao – May 10, 2012

Contents of the talk

- Density-dependent incompressible Euler system
  - (i) well-posedness in endpoint Besov spaces
  - (ii) propagation of geometric structures
- Hyperbolic operators with low regularity coefficients
  - (i) history of the problem
  - (ii) coefficients log-Zygmund in t, log-Lipschitz in x

NON-HOMOGENEOUS INCOMPRESSIBLE INVISCID FLUIDS

Well-posedness in endpoint Besov spaces

Introduction

$$(DDE) \qquad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho \left(\partial_t u + u \cdot \nabla u \right) + \nabla \Pi = \rho f \\ \operatorname{div} u = 0 \end{cases}$$

$(t,x) \in [0, T_0] \times \mathbb{R}^N$

- $\rho(t,x) \in \mathbb{R}_+$ density of the fluid
- $u(t,x) \in \mathbb{R}^N$ velocity field
- $\Pi(t,x) \in \mathbb{R}$ pressure of the fluid
- $f(t,x) \in \mathbb{R}^N$ body force (given)

(i) Coupling of transport equations by $u \implies$ no gain of regularity

(ii) Preserving regularity $\implies B^s_{p,r} \hookrightarrow \mathcal{C}^{0,1}$

Littlewood-Paley Theory, Besov spaces

Dyadic partition of unity in phase-space:

$$\chi_{-1}(\xi) + \sum_{\nu=0}^{+\infty} \psi_\nu(\xi) \equiv 1$$

$$\operatorname{supp} \chi_{-1} \subset \{|\xi| \leq 1\} , \quad \operatorname{supp} \psi_\nu \subset \left\{2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\right\}$$

$\Longrightarrow$ Operators:

$$\Delta_{-1} := \chi_{-1}(D_x) , \quad \Delta_\nu := \psi_\nu(D_x) , \quad S_\nu := \sum_{j=-1}^{\nu-1} \Delta_j$$

$$\Rightarrow \quad \forall u \in \mathcal{S}'(\mathbb{R}^N) , \qquad u = \sum_{\nu=-1}^{+\infty} \Delta_\nu u$$

$\Longrightarrow$ **Besov space $\mathbf{B^s_{p,r}}$** : $u \in \mathcal{S}'$ such that

$$\|u\|_{B^s_{p,r}} := \left\| \left(2^{s\nu} \|\Delta_\nu u\|_{L^p}\right)_{\nu \geq -1} \right\|_{\ell^r} < +\infty$$

Previous results

- Various well-posedness results under different hypothesis (Itoh, Zhou, Danchin...)
- Danchin (2010)
  (i) well-posedness in $B^s_{p,r} \hookrightarrow \mathcal{C}^{0,1}$, $1 < p < +\infty$, with $u_0 \in L^2$
  (ii) $p = +\infty$ with $\rho_0 \sim \overline{\rho}$ (cst)
- Now, functional framework: $B^s_{\infty,r}$, with

$$s > 1 \,,\; r \in [1,+\infty] \qquad \text{or} \qquad s = 1 \,,\; r = 1$$

  ▷ $\mathcal{C}^s \equiv B^s_{\infty,\infty}$
  ▷ $B^1_{\infty,1}$ (the largest space embedded in $\mathcal{C}^{0,1}$)

Finite energy data

- Initial density:

$$\rho_0 \in B^s_{\infty,r} \qquad \text{such that} \qquad 0 < \rho_* \le \rho_0 \le \rho^*$$

- Initial velocity field:

$$u_0 \in B^s_{\infty,r} \qquad \text{with} \qquad \operatorname{div} u_0 = 0$$

- External force: $f \equiv 0$ (for simplicity)

Theorem (Danchin, F. – 2011)

Suppose $u_0 \in L^2$.
Then there exists a unique local solution $(\rho, u, \nabla\Pi)$ on $[0,T] \times \mathbb{R}^N$:

- $\rho \in \mathcal{C}([0,T]; B^s_{\infty,r})$ and bounded away from 0,
- $u \in \mathcal{C}([0,T]; B^s_{\infty,r}) \cap \mathcal{C}^1([0,T]; L^2)$,
- $\nabla\Pi \in L^1([0,T]; B^s_{\infty,r}) \cap \mathcal{C}([0,T]; L^2)$.

Continuation criterion

Theorem (Danchin, F. – 2011)

Let $(\rho, u, \nabla\Pi)$ be a solution to (DDE) on $[0, T^*[\times\mathbb{R}^N$, and suppose

$$\int_0^{T^*} \left(\|\nabla u\|_{L^\infty} \;+\; \|\nabla\Pi\|_{B^{s-1}_{\infty,r}}\right) dt \;<\; +\infty\,.$$

If $T^* < +\infty$, then $(\rho, u, \nabla\Pi)$ can be continued beyond T^*.

- ▷ The lifespan in $B^s_{\infty,r}$ $(s > 1)$ is the same as the lifespan in $B^1_{\infty,1}$
- ▷ If $s > 1$, then $\|\nabla u\|_{L^\infty}$ can be replaced by $\|\Omega\|_{L^\infty}$

- Proof based on refined estimates
  (Bony's paraproduct decomposition)

Other results

- Infinite energy data

  $u_0 \in L^p$ and $\nabla u_0 \in L^q$, with $1/p + 1/q \geq 1/2$

- On the lifespan of the solution

  (i) $(\rho, u, \nabla\pi)$ solution whose lifespan T^*, and

$$\left(\widetilde{\rho}\,,\,\widetilde{u}\,,\,\nabla\widetilde{\Pi}\right)(t,x) \;:=\; \left(\rho\,,\,\varepsilon\,u\,,\,\varepsilon^2\,\nabla\Pi\right)(\varepsilon\,t,x)$$

$$\Longrightarrow \qquad \widetilde{T}^* \geq T^*/\varepsilon$$

  (ii) If $N = 2$, then

$$T^* \;\geq\; \frac{C}{\|u_0\|_{L^2\cap B^1_{\infty,1}}}\,\log\left(1 \,+\, C\,\log\frac{1}{\|\nabla\rho_0\|_{B^0_{\infty,1}}}\right)$$

Propagation of geometric structures

On the vorticity

- **Vorticity** $\Omega := \nabla u - {}^t\nabla u$
  (i) *Biot-Savart Law:* $u^i = -(-\Delta)^{-1} \sum_{j=1}^{N} \partial_j \Omega_{ij}$
  (ii) *Vorticity equation:*

$$\partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega + \nabla \left(\frac{1}{\rho} \right) \wedge \nabla \Pi = 0$$

- Results for homogeneous fluids
  - $N = 2$: vortex patches structure, i.e. $\omega_0 = \chi_{D_0}$
    $\Longrightarrow \exists !$ global solution (*Yudovich Theorem*) and $\omega(t) = \chi_{D(t)}$
    Majda (1986); Chemin (1991, 1993); Danchin (1997)
  - $N \geq 3$: striated and conormal regularity
    Gamblin, Saint-Raymond (1995); Danchin (1999)

Definitions

$$\varepsilon \in]0,1[\,, \quad \eta \in [\varepsilon, 1+\varepsilon]\,, \quad f \in L^\infty$$

- **Striated regularity**

  Vector-field $X \in \mathcal{C}^\varepsilon(\mathbb{R}^N)$ such that $\operatorname{div} X \in \mathcal{C}^\varepsilon(\mathbb{R}^N)$

$$f \in \mathcal{C}^\eta_X \quad \overset{def}{\Longleftrightarrow} \quad \partial_X f \in \mathcal{C}^{\eta-1}(\mathbb{R}^N)$$

- **Conormal regularity**

  $\Sigma \subset \mathbb{R}^N$ compact hypersurface of class $\mathcal{C}^{1,\varepsilon}$

  $\mathcal{T}^\varepsilon_\Sigma := \{X \in \mathcal{C}^\varepsilon,\ \operatorname{div} X \in \mathcal{C}^\varepsilon,\ X \text{ tangent to } \Sigma\}$

$$f \in \mathcal{C}^\eta_\Sigma \quad \overset{def}{\Longleftrightarrow} \quad \forall\, X \in \mathcal{T}^\varepsilon_\Sigma\,, \quad \partial_X f \in \mathcal{C}^{\eta-1}(\mathbb{R}^N)$$

General hypothesis

- External force: $f \equiv 0$
- Initial velocity field and its vorticity:

$$u_0 \in L^p\,, \quad \Omega_0 \in L^\infty \cap L^q \quad \text{with} \quad 1/p + 1/q \geq 1/2$$

- Initial density:

$$\rho_0 \in W^{1,\infty} \qquad \text{such that} \qquad 0 < \rho_* \leq \rho_0 \leq \rho^*$$

- Geometric assumptions:

$$\Omega_0\,,\ \nabla\rho_0 \in \mathcal{C}^\varepsilon_{X_0} \qquad \text{or} \qquad \Omega_0\,,\ \nabla\rho_0 \in \mathcal{C}^\varepsilon_{\Sigma_0}$$

- **Flow associated to u**:

$$\psi_{\mathbf{t}}(\mathbf{x}) := \mathbf{x} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{u}\left(\tau, \psi_\tau(\mathbf{x})\right) \mathbf{d}\tau$$

Results

Theorem (F. – 2012)

There exists a unique local solution $(\rho, u, \nabla\Pi)$ on $[0,T] \times \mathbb{R}^N$:

- $\rho \in L^\infty([0,T]; W^{1,\infty})$, with $\rho_* \leq \rho \leq \rho^*$;
- $u \in \mathcal{C}([0,T]; L^p) \cap L^\infty([0,T]; \mathcal{C}^{0,1})$ and $\Omega \in \mathcal{C}([0,T]; L^q)$;
- $\nabla\Pi \in \mathcal{C}([0,T]; L^2)$, with $\nabla^2\Pi \in L^\infty([0,T]; L^\infty)$.

Moreover, striated regularity is preserved: uniformly on $[0,T]$,

$$\nabla\rho(t)\,,\ \Omega(t) \ \in \mathcal{C}^{\varepsilon}_{X_t} \qquad \text{and} \qquad u(t)\,,\ \nabla\Pi(t) \ \in \mathcal{C}^{1+\varepsilon}_{X_t}\,,$$

where $X_t(x) := (\partial_{X_0(x)}\psi_t)(\psi_t^{-1}(x))\,.$

- ▷ Analogous result for conormal regularity, with $\Sigma_t := \psi_t(\Sigma_0)$
- ▷ Lower bound for the lifespan
- ▷ "Hölder continuous vortex patches"

WAVE EQUATIONS WITH LOW REGULARITY COEFFICIENTS

History of the problem

Setting of the problem

$$Lu \;:=\; \partial_t^2 u \;-\; \sum_{j,k=1}^{N} \partial_j \Big(a_{jk}(t,x)\, \partial_k u \Big)$$

on a strip $[0,T] \times \mathbb{R}^N$, with

$$0 \;<\; \lambda_0 \, |\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\, \xi_j\, \xi_k \;\leq\; \Lambda_0 \, |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0\}$$

▷ **Aim:** studying the Cauchy problem

$$(CP) \qquad \begin{cases} Lu \;=\; f \\ u_{|t=0} \;=\; u_0 \,, \quad \partial_t u_{|t=0} \;=\; u_1 \end{cases}$$

in the Sobolev spaces framework

Classical result

$$a_{jk}(t,x) \quad \begin{cases} \text{Lipschitz continuous in } t \\ \text{only measurable with respect to } x \end{cases}$$

$\Longrightarrow$ energy estimate with **no loss of derivatives**:

$$\sup_{0\le t\le T}\left(\|u(t,\cdot)\|_{H^{s+1}} + \|\partial_t u(t,\cdot)\|_{H^s}\right) \le$$

$$\le C_s\left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Lu(t,\cdot)\|_{H^s}\, dt\right)$$

$\Longrightarrow$ well-posedness of (CP) in $H^{s+1}\times H^s$

General idea:

- ▷ lower regularity assumptions with respect to t
- ▷ suitable hypothesis on x to compensate it

Coefficients depending only on time

$$Lu(t,x) \ := \ \partial_t^2 u(t,x) \ - \ \sum_{j,k=1}^{N} a_{jk}(t)\ \partial_j \partial_k u(t,x)$$

- *Integral log-Lipschitz condition*

  Colombini, De Giorgi, Spagnolo (1979)

$$\int_0^{T-\tau} \left| a_{jk}(t+\tau) \ - \ a_{jk}(t) \right| dt \ \leq \ C_0 \ \tau \ \log\left(1 + \frac{1}{\tau}\right)$$

- *Integral log-Zygmund condition*

  Tarama (2007)

$$\int_\tau^{T-\tau} \left| a_{jk}(t+\tau) + a_{jk}(t-\tau) - 2a_{jk}(t) \right| dt \ \leq \ C_0 \ \tau \ \log\left(1 + \frac{1}{\tau}\right)$$

Theorem

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1-\delta}}+\|\partial_t u(t,\cdot)\|_{H^{s-\delta}}\right)\leq$$
$$\leq C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^{s}}+\int_0^T\|Lu(t,\cdot)\|_{H^s}\,dt\right)$$

- Energy estimates with a **fixed loss of derivatives**
- (CP) is well-posed in $H^\infty(\mathbb{R}^N)$, globally in time
- Proof
  - ▷ Fourier transform
  - ▷ smoothing out the coefficients by a convolution kernel
  - ▷ linking approximation parameter and dual variable: $\varepsilon = |\xi|^{-1}$

Coefficients depending on (t,x)

$$Lu \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j\Big(a_{jk}(t,x)\,\partial_k u\Big)$$

with *pointwise log-Lipschitz condition* (Colombini, Lerner (1995))

$$\sup_{(t,x)}\big|a_{jk}(t+\tau,x+y)\,-\,a_{jk}(t,x)\big| \;\leq\; C_0\;(\tau+|y|)\;\log\left(1+\frac{1}{\tau+|y|}\right)$$

Theorem

$$\sup_{0\leq t\leq T^*}\left(\|u(t,\cdot)\|_{H^{s+1-\beta t}}+\|\partial_t u(t,\cdot)\|_{H^{s-\beta t}}\right)\;\leq$$

$$\leq\; C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^{s}}+\int_0^{T^*}\|Lu(t,\cdot)\|_{H^{s-\beta t}}\,dt\right)$$

Log-Zygmund condition for coefficients depending both on t and x

The scalar case: $N = 1$

- Log-Zygmund condition in time:

$$\sup_{(t,x)} \left| a_{jk}(t+\tau,x) + a_{jk}(t-\tau,x) - 2a_{jk}(t,x) \right| \leq C_0\, \tau \log\left(1+\frac{1}{\tau}\right)$$

- Log-Lipschitz condition in space:

$$\sup_{(t,x)} \left| a_{jk}(t,x+y) - a_{jk}(t,x) \right| \leq C_0 |y| \log\left(1+\frac{1}{|y|}\right)$$

Theorem (Colombini, Del Santo – 2009)

$$\sup_{0\leq t\leq T^*} \left(\|u(t,\cdot)\|_{H^{s+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{s-\beta t}} \right) \leq$$

$$\leq C_s \left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^{T^*} \|Lu(t,\cdot)\|_{H^{s-\beta t}}\, dt \right)$$

Remarks

- Energy estimate with a **loss of derivatives increasing in time**
- $s \in]-1/2, 0[$
- Complete operator (Colombini, F. – 2010):
  - ▷ $b \cdot \nabla_{(t,x)} u$, with $b \in L^\infty_T(\mathcal{C}^\omega(\mathbb{R}^N))$
  - ▷ $c\, u$, with $c \in L^\infty([0,T] \times \mathbb{R}^N)$

$$\Longrightarrow \qquad s > -\min\left\{\frac{1}{2}, \frac{\omega}{1+\log 2}\right\}$$

- $a_{jk} \in \mathcal{C}^\infty_b(\mathbb{R}^N) \quad \Longrightarrow \quad$ well-posedness in H^∞, global in time

The general case: $N \geq 1$

- Log-Zygmund condition in time:

$$\sup_{(t,x)} \left| a_{jk}(t+\tau,x) + a_{jk}(t-\tau,x) - 2a_{jk}(t,x) \right| \;\leq\; C_0\,\tau\,\log\left(1+\frac{1}{\tau}\right)$$

- Log-Lipschitz condition in space:

$$\sup_{(t,x)} \left| a_{jk}(t,x+y) \;-\; a_{jk}(t,x) \right| \;\leq\; C_0 |y| \log\left(1+\frac{1}{|y|}\right)$$

Theorem (Colombini, Del Santo, F., Métivier – 2012)

$$\sup_{0\leq t\leq T^*} \left(\|u(t,\cdot)\|_{H^{s+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{s-\beta t}} \right) \leq$$

$$\leq\; C_s \left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^{T^*} \|Lu(t,\cdot)\|_{H^{s-\beta t}}\,dt \right)$$

On the proof

- *Paradifferential calculus with parameters*
  - $a(t,x,\xi,\gamma)$, rough in x $\longmapsto$ classical symbol $\sigma_a(t,x,\xi,\gamma)$
  - Paradifferential operator: $T_a := \sigma_a(t,x,D_x,\gamma)$
  - $a \geq \lambda(\gamma + |\xi|)^m > 0 \implies T_a =$ positive operator
- Approximation of the coefficients (only in time)

$$a_{jk} \rightsquigarrow a_{jk,\varepsilon} \qquad (\text{ and then } \qquad \varepsilon = 2^{-\nu})$$

- Symbol of order 0

$$\alpha := (\gamma^2 + |\xi|^2)^{-1/2} \left(\sum_{j,k=1}^{N} a_{jk,\varepsilon}(t,x)\,\xi_j\,\xi_k + \gamma^2 \right)^{-1/2}$$

- Localized energy technique

THANK YOU !