Decay rates of magnetoelastic waves in an unbounded conductive medium

by

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Abstract

We study the uniform decay of the total energy of solutions of the system of magnetoelasticity with localized damping "near" infinity in an exterior 3-D domoin. Using appropiatte multipliers and recent work due to R. Charão and R. Ikekata (Nonlinear Analysis, 2007) we conclude that the energy decays like $(1+t)^{-1}$ as $t \to +\infty$.

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1 Introduction

The model we consider in this work is motivated by a phenomenon which appears frequently in nature: The interaction between the strain and electromagnetic fields in an elastic body. In the middle 1950's L. Knopoff [9] investigated the propagation of elastic waves in the presence of Earth's magnetic field. The correspondig model nowdays is part of the theory of magnetoelastic waves. More complete models were studied by J. Dunkin and A. Eringen [5] where they considered the propagation of elastic waves under the influence of both, a magnetic and an electric field. Further recent results on the subject can be found in [2], [12], [15] and the references therein.

The system under consideration in this work may be viewed as a coupling between the hyperbolic system of elastic waves and a parabolic system for the magnetic field. Somehow, the coupled system under consideration has the same structure as the classical isotropic thermoelastic system (see [13], [16] and references therein). When we try to study the asymptotic behavior of the total energy as $t \to +\infty$ for the model we are considering in this work, then, similar difficulties as for the thermoelastic system (in $n = 2$ or 3 dimensions) will appear as clearly were pointed out in [6], [12] or [14].

In [2] the model of magnetoelasticity was considered in a bounded region Ω of $\mathbb{R}^3$ with an extra localized dissipation $\rho(x, u_t)$ effective only on a little "piece" of Ω. The conclusion in [2] was that the total energy decays uniformly as $t \to +\infty$ provided $\rho(x, u_t)$ satisfied suitable conditions. The decay was exponential if ρ behaved "almost" linear in u_t and polynomially if ρ had an "almost" polynomial growth.

In this work we consider the magnetoelastic system in the exterior of a compact body and we replace $\rho(x, u_t)$ by a localized dissipation "near infinity". The final result is that the total energy associated to the system decays like $(1 + t)^{-1}$ as $t \to +\infty$.

Let us describe the model we will consider in this work: Consider a compact set $\mathcal{O}$ of $\mathbb{R}^3$ which is star-shaped with respect to the origem $(0, 0, 0) \in \mathcal{O}$, that is $\eta(x) \cdot x \geq 0$ for all $x \in \partial\mathcal{O}$ where $\eta(x)$ denotes the unit normal vector at x pointing the exterior of $\mathcal{O}$. Along the next sections we will assume the following hypotheses:

(H1). Let $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ be the exterior domain and $\alpha\colon \Omega \to \mathbb{R}^+$ be a function in $L^\infty(\Omega)$ which is effective only near infinity, that is, there exist constant $a_0 > 0$ and $L >> 1$ such that $\alpha(x) \geq a_0 > 0$ for all $x \in \Omega$ with $|x| \geq L$ (Thus $\alpha(x)$ could be zero for $|x| < L$).

(H2). The boundary of Ω denoted by $\partial\Omega$ is smooth, say of class C^2. Although, for several estimates is enough to assume that $\partial\Omega$ is Lipschitz continuous (see [8]). The model we consider in our discussion is the following.

$$\begin{cases} u_{tt} - a^2\Delta u - (b^2 - a^2)\nabla \operatorname{div} u - \mu_0[\operatorname{curl} h] \times \widetilde{H} + \alpha(x)u_t = 0 \\ \gamma h_t + \operatorname{curl}\operatorname{curl} h - \gamma \operatorname{curl}(u_t \times \widetilde{H}) = 0 \\ \operatorname{div} h = 0 \end{cases} \tag{1.1}$$

in $\Omega \times (0, +\infty)$ with boundary conditions

$$\begin{cases} u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ h \cdot \eta = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ [\operatorname{curl} h] \times \eta = 0 & \text{on } \partial\Omega \times (0, +\infty) \end{cases} \tag{1.2}$$

and initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad h(x, 0) = h_0(x) \text{ in } \Omega. \tag{1.3}$$

In (1.1), the vector field $u = (u_1, u_2, u_3)$ denotes the displacement while $h = (h_1, h_2, h_3)$ denotes the magnetic field. A known magnetic field taken along the x_3-axis is denoted by $\widetilde{H}$. By simplicity we write $\widetilde{H} = (0, 0, 1)$.

All other parameters stand as follows: a and b (both strictly positive) are the Lamé constants with $b^2 > a^2$, μ_0 is the magnetic permeability ($\mu_0 > 0$) and γ is a parameter which is proportional to the electric conductivity ($\gamma > 0$).

A justification for the modelling of the coupled system (1.1) can be found in [11], Chapter 2. In (1.1) and (1.2) we use the standard notation, $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$, $u_{tt} = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_3}{\partial t}\right)$, Δ denotes the Laplacian operator, ∇ is the gradient operator and div the (spatial) divergence, "x" denotes the usual vector product and curl is the rotational operator.

The total energy $E(t)$ associated to system (1.1)–(1.3) is given by

$$E(t) = \frac{1}{2}\int_\Omega \left[|u_t|^2 + a^2|\nabla u|^2 + (b^2 - a^2)(\operatorname{div} u)^2 + \mu_0|h|^2\right] dx \tag{1.4}$$

where

$$|u_t|^2 = \sum_{j=1}^{3}\left|\frac{\partial u_j}{\partial t}\right|^2, \quad |\nabla u|^2 = \sum_{j=1}^{3}|\nabla u_j|^2 \quad \text{and} \quad |h|^2 = \sum_{j=1}^{3} h_j^2$$

Formally, by taking the inner product of the first equation in (1.1) by u_t and the second by h, adding the resulting relations and integrating over Ω we can verify the identity

$$\frac{dE}{dt} = -\frac{\mu_0}{\gamma}\int_\Omega |\operatorname{curl} h|^2 dx - \int_\Omega \alpha(x)|u_t|^2 dx \tag{1.5}$$

which says that the energy $E(t)$ decreases along trajectories.

This paper is devoted to study the large time behavior of the total energy under suitable assumptions on the "localized" extra damping coefficient $\alpha(x)$. Our main result says that $E(t)$ decays like $O((1+t)^{-1})$ as $t \to +\infty$. This is a new result for the coupled system of magnetoelasticity for exterior domains in $\mathbb{R}^3$.

There are several articles dealing with the asymptotic behavior of the total energy associated with system (1.1)–(1.3) in bounded domains (see [2], [6], [12], [14], [15] and the references therein). As far as we know, the only article on system (1.1) in unbounded domain is due to E. Andreou and G. Dassios [1] where they studied the Cauchy problem in $\Omega = \mathbb{R}^3$ (and $\alpha(x) \equiv 0$). They proved that smooth solutions which vanish as $|x| \to +\infty$ do decay (in time) polynomially as $t \to +\infty$. In Section 2 we briefly prove the well posedness of problem (1.1)–(1.3) in appropiatte Hilbert spaces and finally in Section 3 we prove the main result concerning the polynomial decay of the total energy. Our main tools are the use of multipliers and recent ideas introduced by R. Charão and R. Ikekata [3] while dealing with semilinear elastic waves in exterior domains.

Our notations are standard and follow the books [10] and [4].

2 Well posedness: Functional setting

With the notations given in Section 1 we consider the unbounded operator A given as

$$A = \begin{bmatrix} 0 & I & 0 \\ -A_1 & A_3 & B \\ 0 & C & -A_2 \end{bmatrix} \tag{2.1}$$

where $A_1 = -a^2\Delta - (b^2 - a^2)\nabla \operatorname{div}$, $A_2 = \frac{1}{\gamma} \operatorname{curl} \operatorname{curl}$, $A_3 = -\alpha(x)I$, $B = \mu_0[\operatorname{curl}(\cdot)] \times \widetilde{H}$ and $C = \mu_0 \operatorname{curl}(\cdot \times \widetilde{H})$.

Let $H_0^1(\Omega)$ denote the usual Sobolev space and consider the Hilbert space

$$H = [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3 \times W$$

where W is the closure of $\{f \in [C_0^\infty(\Omega)]^3 \text{ such that } \operatorname{div} f = 0 \text{ in } \Omega\}$ in $[L^2(\Omega)]^3$. The norm in $[L^2(\Omega)]^3$ and W is given by $||f||^2 = \int_\Omega |f|^2 dx$, while in $[H_0^1(\Omega)]^3$ the norm is

$$||f||^2_{[H_0^1(\Omega)]^3} = \int_\Omega [a^2|\nabla f|^2 + (b^2 - a^2)|\operatorname{div} f|^2] dx.$$

By Korn's inequality this norm is equivalent to the one induced by $[H^1(\Omega)]^3$.

Using the above notation, problem (1.1)–(1.3) can be rewritten as

$$\begin{aligned} \frac{dU}{dt} &= AU \\ U(0) &= U_0 \end{aligned} \tag{2.2}$$

where

$$U_0 = (u_0, u_1, \mu_0 h_0) \in H \quad \text{and} \quad U(t) = (u, u_t, \mu_0 h).$$

The operators which appear in (2.1) have their domain defined as follows

$$\begin{aligned} \mathcal{D}(A_1) &= [H^2(\Omega)]^3 \cap [H_0^1(\Omega)]^3 \\ \mathcal{D}(A_2) &= \{h \in [H^2(\Omega)]^3 \cap W \text{ such that } [\operatorname{curl} h] \times \eta = 0 \text{ on } \partial\Omega\} \\ \mathcal{D}(A_3) &= [H_0^1(\Omega)]^3 \\ \mathcal{D}(B) &= \{h \in W \text{ such that } [\operatorname{curl} h] \times \widetilde{H} \in [L^2(\Omega)]^3\} \\ \mathcal{D}(C) &= \{v \in [L^2(\Omega)]^3 \text{ such that } \operatorname{curl}(v \times \widetilde{H}) \in W\}. \end{aligned}$$

We set

$$\mathcal{D}(A) = [H^2(\Omega)]^3 \cap [H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3 \times \mathcal{D}(A_2).$$

Clearly $\mathcal{D}(A)$ is dense in H. It is convenient to consider the operator $\widetilde{A}$ with domain $\mathcal{D}(\widetilde{A}) = \mathcal{D}(A)$ and given by

$$\widetilde{A} = \begin{bmatrix} 0 & I & 0 \\ -A_1 - I & A_3 & B \\ 0 & C & -A_2 \end{bmatrix}.$$

Lemma 2.1. *With the above notations, the operator $\widetilde{A}$ is dissipative and Image $(I - \widetilde{A}) = H$.*

Proof: Let us consider the natural inner product in $[H^1(\Omega)]^3$

$$\langle u, v\rangle_{[H^1(\Omega)]^3} = a^2(\nabla u, \nabla v) + (b^2 - a^2)(\operatorname{div} u, \operatorname{div} v) + (u, v)$$

where $(\nabla u, \nabla v) = \sum_{i=1}^{3} (\nabla u^i, \nabla v^i)$, $u = (U^1, u^2, u^3)$ and $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. Whenever $U = (u, v, h)$ and $V = (\tilde{u}, \tilde{v}, \tilde{h})$ belong to H we consider the inner product given by

$$\langle U, V\rangle_H = \langle u, \tilde{u}\rangle_{[H^1(\Omega)]^3} + (v, \tilde{v}) + (h, \tilde{h}).$$

Let $U=(u,v,h)\in\mathcal{D}(\widetilde{A})$, we want to calculate $\langle\widetilde{A}U,U\rangle_H$. Using a result due to D. Sheen [17] we know that $(Cv,h)=-(Bh,v)$ holds whenever $(v,h)\in[H_0^1(\Omega)]^3\times\mathcal{D}(A_2)$ therefore

$$\begin{aligned}\langle\widetilde{A}U,U\rangle &= \langle v,u\rangle_{[H^1(\Omega)]^3}+(-A_1u-u+A_3v+Bh,v)+(Cv-A_2h,h)\\ &= \langle v,u\rangle_{[H^1(\Omega)]^3}-(u,v)+(-A_1u+A_3v,v)+(-A_2h,h)\\ &= \langle v,u\rangle_{[H^1(\Omega)]^3}-(u,v)+a^2(\Delta u,v)+(b^2-a^2)(\nabla\operatorname{div}u,v)\\ &\quad-(\alpha(x)v,v)-\frac{1}{\gamma}(\operatorname{curl}\operatorname{curl}h,h).\end{aligned}$$

Using the definition of inner product in H followed by integration by parts we obtain

$$\begin{aligned}\langle\widetilde{A}U,U\rangle_H &= -||\sqrt{\alpha}v||^2-\frac{1}{\gamma}(\operatorname{curl}\operatorname{curl}h,h)\\ &= -||\sqrt{\alpha}v||^2-\frac{1}{\gamma}||\operatorname{curl}h||^2\le 0\end{aligned}$$

because if $h\in\mathcal{D}(A_2)$ then $(h,\operatorname{curl}\operatorname{curl}h)=||\operatorname{curl}h||^2$ (see D. Sheen [17]).

Let $F=(f_1,f_2,f_3)$ any element in H. We need to prove the existence of $U=(u,v,h)\in\mathcal{D}(\widetilde{A})$ such that $U-\widetilde{A}U=F$ which means that

$$\begin{aligned}&u-v=f_1\\ &A_1u+u+(I-A_3)v-Bh=f_2\\ &-Cv+(I+A_2)h=f_3.\end{aligned}\tag{2.3}$$

Since $v=u-f_1$, (2.3) reduces to solve the system

$$\begin{aligned}&A_1u+2u-A_3u-Bh=g_1\\ &-Cu+h+A_2h=g_2\end{aligned}\tag{2.4}$$

where $g_1=f_2+(I-A_3)f_1$ and $g_2=f_3-Cf_1$. Observe that Cf_1 belongs to W because $f_1\in[H_0^1(\Omega)]^3$. Furthermore, since $[H_0^1(\Omega)]^3$ is continuously embedded into the domain of C it follows that $g_2\in W$. Thus the right hand side of system (2.4) belongs to $[L^2(\Omega)]^3\times W$.

Let us consider the space $\mathcal{H}=[H_0^1(\Omega)]^3\times V$ where V is given by

$$V=\{v\in W,\quad \operatorname{curl}v\in[L^2(\Omega)]^3\}.$$

The inner produt in given as follows: For any v and $\tilde{v}$ belonging to V then

$$(v,\tilde{v})_V=\int_\Omega[v\cdot\tilde{v}+\frac{1}{\gamma}\operatorname{curl}v\cdot\operatorname{curl}\tilde{v}]\,dx.$$

We consider the bilinear form $a(\cdot,\cdot)$ on $[H_0^1(\Omega)]^3\times[H_0^1(\Omega)]^3$ given by

$$\begin{aligned}a(u,\tilde{u})=\int_\Omega&[(2+\alpha(x))u\cdot\tilde{u}+a^2\nabla u\cdot\nabla\tilde{u}\\ &+(b^2-a^2)\operatorname{div}u\operatorname{div}\tilde{u}]\,dx.\end{aligned}\tag{2.5}$$

Next, we define the bilinear form $\tilde{B}(\cdot,\cdot)$ on the space $\mathcal{H}$, given by

$$\begin{aligned}&\tilde{B}((u,h),(\tilde{u},\tilde{h}))\\&=a(u,\tilde{u})+(h,\tilde{h})_V-(Bh,\tilde{u})-(Cu,\tilde{h})\end{aligned} \tag{2.6}$$

where B and C are the operators defined at the begining of the section.

We clain that $\tilde{B}$ is continuous and coercive on $\mathcal{H}$. In fact

$$\begin{aligned}|\tilde{B}((u,h),(\tilde{u},\tilde{h}))| &\leq |a(u,\tilde{u})|+|(h,\tilde{h})_V|+|(Bh,\tilde{u})|+|(Cu,\tilde{h})|\\&\leq K_1||u||_{H^1}||\tilde{u}||_{H^1}+||h||_V||\tilde{h}||_V\\&\quad+||Bh||_{L^2}||\tilde{u}||_{H^1}+||Cu||_{L^2}||\tilde{h}||_V\\&\leq K_1||u||_{H^1}||\tilde{u}||_{H^1}+||h||_V||\tilde{h}||_V+\mu_0||\operatorname{curl} h||_{L^2}||\tilde{u}||_{H^1}\\&\quad+K_2||Cu||_{L^2}||\tilde{h}||_V\\&\leq K_3||(u,h)||_{\mathcal{H}}||(\tilde{u},\tilde{h})||_{\mathcal{H}}\end{aligned} \tag{2.7}$$

because $||\operatorname{curl} h||_{L^2}\leq ||h||_V$. We used the notation $||\cdot||_{H^1}$ instead of $||\cdot||_{[H^1(\Omega)]^3}$ and $||\cdot||_{L^2}$ instead of $||\cdot||_{[L^2(\Omega)]^3}$ in order to simplify excess of symbols. In the above inequalities K_1, K_2 and K_3 are positive constant which depend on the Lamé coefficients and/or $||\alpha(\cdot)||_{L^\infty}$. By (2.7) it follows that $\tilde{B}$ is continuous. finally we observe that

$$\begin{aligned}\tilde{B}((u,h),(u,h)) &= a(u,u)+(h,h)_V\\&\geq K_4[||u||^2_{[H^1]^3}+||h||^2_V]\end{aligned}$$

where $K_4>0$ is a positive constant. Again, we used the identity $(Cu,h)=-(Bh,u)$. It follows by Lax-Milgram's lemma that there exists a unique $(u,h)\in\mathcal{H}$ such that

$$\tilde{B}((u,h)(\tilde{u},\tilde{h}))=(g_1,\tilde{u})+(g_2,\tilde{h}) \tag{2.8}$$

for any $(\tilde{u},\tilde{h})\in\mathcal{H}$. It is easy to see that (u,h) in (2.8) is a weak solution of (2.4). Furthermore, since $h\in V$ we know that $Bh\in[L^2(\Omega)]^3$. Thus, u is a weak solution of

$$A_1u+2u=A_3u+Bh+g_1\in[L^2(\Omega)]^3. \tag{2.9}$$

Since A_1 is a strongly elliptic operator it follows from (2,9) and elliptic regularity in an exterior domain that $u\in\mathcal{D}(A_1)=[H^2(\Omega)]^3\cap[H^1_0(\Omega)]^3$. Remains to prove that $h\in\mathcal{D}(A_2)$. Similar procedure as above shows that $h\in[H^2(\Omega)]^3\times W$. We claim that $\operatorname{curl} h\times\eta=0$ on $\partial\Omega$. In fact, taking $\tilde{u}$ in (2.8) and using (2.6) we obtain the identity

$$(h,\tilde{h})_V-(Cu,\tilde{h})=(g_2,\tilde{h})\quad \text{for all } \tilde{h}\in V.$$

Next using the second equation in (2.3) in the above identity give us

$$(h,\tilde{h})_V - (Cu,\tilde{h}) = (-Cu + h + A_2 h, \tilde{h}) \quad \text{for all } \tilde{h} \in V \tag{2.10}$$

Using Green's formula we integrate by parts the term $(A_2 h, \tilde{h})$ and replace in (2.10) to obtain

$$(h,\tilde{h})_V = (h,\tilde{h}) + \frac{1}{\gamma}(\operatorname{curl} h, \operatorname{curl} \tilde{h}) + \frac{1}{\gamma}\int_{\partial\Omega} (\eta \times \operatorname{curl} h)\cdot \tilde{h}\, dS$$

for all $\tilde{h} \in V$. However, the inner product in V is given by

$$(h,\tilde{h})_V = (h,\tilde{h}) + \frac{1}{\gamma}(\operatorname{curl} h, \operatorname{curl} \tilde{h}).$$

Therefore

$$\int_{\partial\Omega} (\eta \times \operatorname{curl} h)\cdot \tilde{h}\, dS = 0$$

for all $\tilde{h} \in V$, which proves our claim. Since $v = u - f_1 \in [H_0^1(\Omega)]^3$, the proof of Lemma 2.1 is now complete.

Theorem 2.2. *Let Ω and $\alpha(x)$ satisfying hypothesis (H1) and (H2) in Section 1 with μ_0 and γ being positive constants. Let $(u_0, u_1, \mu_0 h_0) \in H$. Then, there exists a unique (weak) solution $\{u, h\}$ of system (1.1)–(1.3) in the class*

$$u \in C([0,+\infty); [H_0^1(\Omega)]^3) \cap C^1([0,+\infty); [L^2(\Omega)]^3)$$

and

$$h \in C([0,+\infty); W).$$

Moreover, if $(u_0, u_1, \mu_0 h_0) \in \mathcal{D}(A)$, then, problem (1.1), (1.2), (1.3) has a unique solution (u, h) in the class

$$u \in C([0,+\infty); [H^2(\Omega) \cap H_0^1(\Omega)]^3) \cap C^1([0,+\infty); [H_0^1(\Omega)]^3) \cap C^2([0,+\infty); [L^2(\Omega)]^3)$$

and

$$h \in C([0,+\infty); \mathcal{D}(A_2)) \cap C^1([0,+\infty); W).$$

Proof: By Lemma 2.1 the operator $\widetilde{A}$ is maximal dissipative with $\mathcal{D}(\widetilde{A})$ dense in H, then, by Lumer—Phillips theorem it follows that $\widetilde{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions. Obviously, this implies in particular that the operator A is also an infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on H. Both conclusions of Theorem 2.2 follow from the theory of semigroups.

3 Asymptotic Behavior

In this section we prove the main result of this article, namely the uniform decay rate of the total energy $E(t)$ given by (1.4) and initial data $(u_0, u_1, \mu_0 h_0) \in H$. Since (1.5) is valid we integrate over $[0,t]$ to obtain

$$E(t) + \frac{\mu_0}{\gamma}\int_0^t\int_\Omega |\operatorname{curl} h|^2 dx\, ds + \int_0^t\int_\Omega \alpha(x)|u_t|^2 dx\, ds = E(0) \tag{3.1}$$

for all $t \geq 0$.

Lemma 3.1 (Identities from multipliers). *Let (u,h) be the solution of system (1.1)–(1.3). Let $\varphi\colon \overline{\Omega} \to \mathbb{R}^3$ be (an auxiliary) a continuous function with $\frac{\partial \varphi}{\partial x_j} \in L^\infty(\Omega)$ $(j = 1,2,3)$. Then, the following identities are valid*

$$\begin{aligned} &\frac{d}{dt}(u_t, u) - ||u_t||^2 + a^2||\nabla u||^2 + (b^2 - a^2)||\operatorname{div} u||^2 \\ &\qquad - \mu_0 \int_\Omega u \cdot (\operatorname{curl} h \times \widetilde{H})dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \alpha(x)|u|^2 dx = 0 \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_t \cdot (\varphi : \nabla u)dx + \frac{1}{2}\int_\Omega \operatorname{div}\varphi[|u_t|^2 - a^2|\nabla u|^2 - (b^2 - a^2)(\operatorname{div} u)^2]dx \\ &+ \sum_{i,j,k=1}^{3}\int_\Omega \left[a^2 \frac{\partial \varphi_j}{\partial x_i}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} + (b^2 - a^2)\frac{\partial \varphi_j}{\partial x_i}\frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_j}\right] dx \\ &- \frac{1}{2}\int_{\partial\Omega}(\varphi \cdot \eta)\left\{a^2\left|\frac{\partial u}{\partial \eta}\right|^2 + (b^2 - a^2)(\operatorname{div} u)^2\right\} dS \\ &+ \int_\Omega \alpha(x)u_t \cdot (\varphi : \nabla u)dx - \mu_0\int_\Omega (\operatorname{curl} h \times \widetilde{H}) \cdot (\varphi : \nabla u)dx = 0 \end{aligned} \tag{3.3}$$

Furthermore, if $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$\psi(x) = \begin{cases} a_0 & \text{if} \quad |x| \leq L \\ \frac{La_0}{|x|} & \text{if} \quad |x| \geq L \end{cases}$$

where a_0 and L are as in Hypotheses (H1). Then,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi(x)u_t\cdot(x:\nabla u)dx+\frac{1}{2}\int_\Omega[3\psi+r\psi_r]\Big\{|u_t|^2\\
&-a^2|\nabla u|^2-(b^2-a^2)(\operatorname{div}u)^2\Big\}dx\\
&-\frac{1}{2}\int_{\partial\Omega}(x\cdot\eta)\psi(r)\Big[a^2\Big|\frac{\partial u}{\partial\eta}\Big|^2+(b^2-a^2)(\operatorname{div}u)^2\Big]dS\\
&+\int_\Omega(b^2-a^2)\Big[\psi(r)(\operatorname{div}u)^2+\frac{\psi_r}{r}\operatorname{div}u(x:\nabla u)\cdot x\Big]dx\\
&+\int_\Omega a^2\Big[\psi(r)|\nabla u|^2+\frac{\psi_r}{r}|x:\nabla u|^2\Big]dx+\int_\Omega\alpha\psi u_t\cdot(x:\nabla u)dx\\
&-\mu_0\int_\Omega(\operatorname{curl}h\times\widetilde{H})\psi(r)\cdot(x:\nabla u)dx=0
\end{aligned}\tag{3.4}$$

where $r=|x|$, $\psi_r=\frac{d\psi}{dr}=\frac{x}{r}\cdot\nabla\psi$ and $\varphi:\nabla u=(\varphi\cdot\nabla u_1,\varphi\cdot\nabla u_2,\varphi\cdot\nabla u_3)$ with $u=(u_1,u_2,u_3)$.

Proof: Taking the inner product of the first equation in (1.1) with u followed by integration over Ω give us identity (3.2). Next we use the multiplier $\varphi:\nabla u$. We take the inner product of the first equation in (1.1) with $\varphi:\nabla u$. Integration of the resulting identity over Ω give us (3.3). Finally (3.4) is obtained from (3.3) by choosing $\varphi(x)=\psi(r)x$.

Lemma 3.2. *Let $\{u,h\}$ be the solution of system (1.1)–(1.3). Consider positive numbers k and ε and define*

$$G_k(t)=\int_\Omega\psi u_t\cdot(x:\nabla u)dx+\varepsilon\int_\Omega u_t\cdot u dx+\frac{\varepsilon}{2}\int_\Omega\alpha(x)|u|^2dx+kE(t).$$

Then, the following estimate

$$\begin{aligned}
&\frac{d}{dt}G_k(t)+\int_\Omega\Big[\frac{3\psi+r\psi_r}{2}-\varepsilon+\frac{k}{2}\alpha(x)\Big]|u_t|^2dx\\
&\quad+\frac{k}{2}\int_\Omega\alpha(x)|u_t|^2dx+\int_\Omega a^2\Big\{\varepsilon-\frac{3\psi+r\psi_r}{2}+\psi+r\psi_t\Big[\frac{1}{2}+\frac{b}{2a}\Big]\Big\}|\nabla u|^2dx\\
&\quad+\frac{k\mu_0}{\gamma}\int_\Omega|\operatorname{curl}h|^2dx\\
&\quad+\int_\Omega(b^2-a^2)\Big\{\varepsilon-\frac{3\psi+r\psi_r}{2}+\psi+r\psi_r\Big[\frac{1}{2}+\frac{b}{2a}\Big]\Big\}(\operatorname{div}u)^2dx\\
&\le-\int_\Omega\alpha(x)\psi u_t\cdot(x:\nabla u)dx+\varepsilon\mu_0\int_\Omega u\cdot(\operatorname{curl}h\times\widetilde{H})dx\\
&\quad+\mu_0\int_\Omega\psi\operatorname{curl}h\cdot(x:\nabla u)dx
\end{aligned}\tag{3.5}$$

holds.

Proof: We adapt to our case some ideas used by Charao and Ikehata [3] for the system of elastic waves in exterior domains. Let us multiply identity (1.5) by $k > 0$ and (3.2) by $\varepsilon > 0$. Adding both resulting identities with (3.4), using hypothesis (H1) and the fact that $\varphi_r \leq 0$ we obtain the following inequality:

$$
\begin{aligned}
&\frac{d}{dt}G_k(t) + \int_\Omega \Big[\frac{3\psi + r\psi_r}{2} - \varepsilon + \frac{k\alpha}{2}\Big]|u_t|^2 dx + \frac{k}{2}\int_\Omega \alpha |u_t|^2 dx \\
&\quad + a^2 \int_\Omega \Big[\varepsilon - \frac{3\psi + r\psi_r}{2} + \psi(r) + r\psi_r\Big] |\nabla u|^2 dx \\
&\quad + \frac{k\mu_0}{\gamma}\int_\Omega |\operatorname{curl} h|^2 dx + (b^2 - a^2)\int_\Omega \Big[\varepsilon + \psi - \frac{3\psi + r\psi_r}{2}\Big](\operatorname{div} u)^2 dx \\
&\quad + (b^2 - a^2)\int_\Omega \frac{\psi_r}{r}\operatorname{div} u (x : \nabla u)\cdot x dx \\
&\leq \varepsilon\mu_0 \int_\Omega u \cdot (\operatorname{curl} h \times \widetilde{H}) dx - \int_\Omega \alpha\psi u_t \cdot (x : \nabla u) dx \\
&\quad + \mu_0 \int_\Omega \psi(\operatorname{curl} h \times \widetilde{H})\cdot (x : \nabla u) dx.
\end{aligned}
\tag{3.6}
$$

Let $\delta = 1 + \frac{b}{a}$. We use Young's inequality to obtain

$$
\begin{aligned}
(b^2 - a^2)|\operatorname{div} u||\nabla u| &\leq (b^2 - a^2)\frac{\delta}{2}|\nabla u|^2 + \frac{1}{2\delta}|\operatorname{div} u|^2 (b^2 - a^2) \\
&= a^2 \frac{(b-a)}{2a}|\nabla u|^2 + \Big(\frac{a+b}{2a}\Big)(b^2 - a^2)(\operatorname{div} u)^2.
\end{aligned}
\tag{3.7}
$$

Since $(\operatorname{div} u)(x : \nabla u)\cdot x \leq |\operatorname{div} u|\, |x|^2\, |\nabla u|$ and $\psi_r \leq 0$. Then multiplying both sides by $\frac{\psi_r}{r}$ we obtain

$$
\frac{\psi_r}{r}\operatorname{div} u (x : \nabla u)\cdot x \geq r\psi_r |\operatorname{div} u|\nabla u|. \tag{3.8}
$$

Thus, using (3.8) and (3.7) in the last integral of the left hand side integral of (3.6) we obtain (3.5).

Lemma 3.3. *Assume $0 < a^2 < b^2 < 4a^2$ and let (u, h) be the solution of system (1.1)–(1.3). If (H1) is valid then there exist positive constants ε and ε_1 such that*

$$
\begin{aligned}
&M\int_0^t E(s)ds + \frac{\varepsilon\mu_0}{2\gamma}||\operatorname{curl} H||^2 \\
&\quad + \Big(\frac{k}{2} - \frac{L^2 a_0^2 ||\alpha||_\infty}{a^2\varepsilon_1}\Big)\int_0^t \int_\Omega \alpha(x)|u_t|^2 dx\, ds \\
&\quad + G_k(t) + \Big(\frac{k\mu_0}{\gamma} - \frac{2\mu_0^2 L^2 a_0^2}{a^2\varepsilon_1}\Big)\int_0^t \int_\Omega |\operatorname{curl} h|^2 dx\, ds \\
&\leq G_k(0) + \varepsilon\mu_0 \int_\Omega [h_0 - \operatorname{curl}(u_0 \times \widetilde{H})]\cdot H\, dx
\end{aligned}
\tag{3.9}
$$

where $M = \min\{\varepsilon\mu_0, \varepsilon_1/4\}$, *k is any positive number such that* $k > \max\left\{\frac{2L^2a_0^2||\alpha||_\infty}{a^2\varepsilon_1}, \frac{2L^2a_0^2\mu_0^2\gamma}{a^2\varepsilon_1}\right\}$ *and* $H(x,t) = \int_0^t h(x,s)ds$.

Proof: Let us define

$$E_1(t) = \frac{1}{2}\int_\Omega \{|u_t|^2 + a^2|\nabla u|^2 + (b^2 - a^2)(\operatorname{div} u)^2\}dx.$$

Let us choose $\varepsilon > 0$ such that $\frac{a_0}{2}(1+\frac{b}{a}) < \varepsilon < \frac{3a_0}{2}$ which is possible because $b^2 < 4a^2$. Next, we can choose $\varepsilon_1 > 0$ such that

$$\frac{3\psi + r\psi_r}{2} - \varepsilon + \frac{k\alpha(x)}{2} \geq \varepsilon_1 > 0 \tag{3.10}$$

and

$$\varepsilon - \frac{3\psi + r\psi_r}{2} + \psi + r\psi_r(\frac{a+b}{2a}) \geq \varepsilon_1 > 0. \tag{3.11}$$

In fact, say for instance $\varepsilon = a_0(1+\frac{b}{4a})$ (then $\frac{a_0}{2}(1+\frac{b}{a}) < a_0(1+\frac{b}{4a}) = \varepsilon < \frac{3}{2}a_0$ because $b < 2a$). Then, we can choose $\varepsilon_1 = a_0(\frac{1}{2} - \frac{b}{4a})$ and it is easy to verify that (3.10) and (3.11) hold for all $x \in \Omega$.

Using (3.10) and (3.11) together with (3.5) we deduce the estimate

$$\begin{aligned}
&\frac{d}{dt}G_k(t) + \varepsilon_1 E_1(t) + \frac{k}{2}\int_\Omega \alpha(x)|u_t|^2dx \\
&\quad + \frac{k\mu_0}{\gamma}\int_\Omega |\operatorname{curl} h|^2dx \\
&\leq -\int_\Omega \alpha(x)\psi u_t\cdot(x:\nabla u)dx + \varepsilon\mu_0\int_\Omega u\cdot(\operatorname{curl} h\times\widetilde{H})dx \\
&\quad + \mu_0\int_\Omega \psi \operatorname{curl} h\cdot(x:\nabla u)dx.
\end{aligned} \tag{3.12}$$

Now, we will estimate two terms on the right hand side of (3.12). Since $|x|\psi(x) \leq La_0$ in $\mathbb{R}^3$, we obtain for any $\varepsilon_2 > 0$

$$\begin{aligned}
\Big|\int_\Omega \alpha(x)u_t\psi\cdot(x:\nabla u)dx\Big| &\leq La_0\int_\Omega \alpha(x)|u_t||\nabla u|dx \\
&\leq \frac{La_0}{\varepsilon_2}\int_\Omega \alpha(x)|u_t|^2dx + \frac{La_0\varepsilon_2}{4}\int_\Omega \alpha(x)|\nabla u|^2dx.
\end{aligned} \tag{3.13}$$

Let us choose $\varepsilon_2 = \frac{a^2\varepsilon_1}{La_0||\alpha||_{L^\infty}}$ to obtain from (3.13) the inequality

$$\begin{aligned}
&\Big|\int_\Omega \alpha(x)\psi u_t\cdot(x:\nabla u)dx\Big| \\
&\leq \frac{L^2a_0^2||\alpha||}{a^2\varepsilon_1}L^\infty\int_\Omega \alpha(x)|u_t|^2dx + \frac{a^2\varepsilon_1}{4}\int_\Omega |\nabla u|^2dx.
\end{aligned} \tag{3.14}$$

Similarly we obtain for any $\delta > 0$

$$\begin{aligned} &\mu_0\Big|\int_\Omega \psi \operatorname{curl} h\cdot(x:\nabla u)dx\Big| \\ &\le \mu_0 L a_0\int_\Omega |\operatorname{curl} h||\nabla u|dx \\ &\le \frac{\mu_0 L a_0}{\delta a^2}\int_\Omega |\operatorname{curl} h|^2dx + \frac{\mu_0 L a_0\delta a^2}{4}\int_\Omega |\nabla u|^2dx. \end{aligned} \tag{3.15}$$

Using (3.14) and (3.15) together with (3.12) and choosing $\delta = \frac{\varepsilon_1}{2\mu_0 L a_0}$ we obtain

$$\begin{aligned} &\frac{d}{dt}G_k(t) + A_k\int_\Omega \alpha(x)|u_t|^2dx + B_k\int_\Omega |\operatorname{curl} h|^2dx \\ &\quad + C\int_\Omega |\nabla u|^2dx + \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2dx + \frac{\varepsilon_1}{2}(b^2-a^2)\int_\Omega (\operatorname{div} u)^2dx \\ &\le \varepsilon\mu_0\int_\Omega u\cdot(\operatorname{curl} h\times \widetilde{H})dx \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} A_k &= \frac{k}{2} - \frac{L^2a_0^2||\alpha||_{L^\infty}}{a^2\varepsilon_1} > 0, \\ B_k &= \frac{k\mu_0}{\gamma} - \frac{\mu_0 L a_0}{\delta a^2} > 0 \quad \text{and} \quad C = \frac{a^2\varepsilon_1}{8}. \end{aligned}$$

Clearly we are choosing k large enough so that A_k and B_k are non-negative.

Finally we consider the field $H(x,t) = \int_0^t h(x,s)ds$. Since $\{u,h\}$ is the solution of problem (1.1)–(1.3) then we can easily verify that $H(x,t)$ satisfies

$$\gamma H_t + \operatorname{curl}\operatorname{curl} H - \gamma\operatorname{curl}(u\times\widetilde{H}) = \gamma[h_0 - \operatorname{curl}(u_0\times\widetilde{H})]. \tag{3.17}$$

Taking the inner product of (3.17) by H_t ad integrating the result over $\Omega\times[0,t]$ we obtain

$$\begin{aligned} &\int_0^t ||h||^2ds + \frac{1}{2\gamma}||\operatorname{curl} H||^2 \\ &= \int_0^t\int_\Omega \operatorname{curl}(u\times\widetilde{H})\cdot h\,dx\,ds + \int_\Omega [h_0 - \operatorname{curl}(u_0\times\widetilde{H})]\cdot H\,dx. \end{aligned} \tag{3.18}$$

Next, we integrate inequality (3.16) over $[0,t]$ and add the result with indentity (3.18) multiplied by $\varepsilon\mu_0$ to obtain

$$\begin{aligned}
&\varepsilon\mu_0\int_0^t ||h||^2 ds + \frac{\varepsilon\mu_0}{2\gamma}||\operatorname{curl} H||^2 + G_k(t)\\
&\quad + A_k\int_0^t\int_\Omega \alpha(x)|u_t|^2 dx\, ds + B_k\int_0^t\int_\Omega |\operatorname{curl} h|^2 dx\, ds\\
&\quad + C\int_0^t\int_\Omega |\nabla u|^2 dx\, ds + \frac{\varepsilon_1}{2}\int_0^t\int_\Omega |u_t|^2 dx\, ds\\
&\quad + \frac{\varepsilon_1}{2}(b^2 - a_2)\int_0^t\int_\Omega (\operatorname{div} u)^2 dx\, ds\\
&\le \varepsilon\mu_0\int_0^t\int_\Omega \operatorname{curl}(u\times\widetilde{H})\cdot h\, dx\, ds\\
&\quad + \varepsilon\mu_0\int_0^t\int_\Omega u\cdot(\operatorname{curl} h\times\widetilde{H})dx\, ds + G_k(0)\\
&\quad + \varepsilon\mu_0\int_\Omega [h_0 - \operatorname{curl}(u_0\times\widetilde{H}]\cdot H\, dx\\
&= G_k(0) + \varepsilon\mu_0\int_\Omega [h_0 - \operatorname{curl}(u_0\times\widetilde{H})]\cdot H dx
\end{aligned} \tag{3.19}$$

because $u\cdot(\operatorname{curl} h\times\widetilde{H}) = -\operatorname{curl}(u\times\widetilde{H})\cdot h$ holds for any three vectors u, $\operatorname{curl} h$ and $\widetilde{H}$.

Lemma 3.4. *Let $0 < a^2 < b^2 < 4a^2$ and $\{u, h\}$ be the solution of system (1.1)–(1.3). Assuming (H1) and (H2) are valid and $h_0 \in W \cap L^2(\Omega)$, then we can find positive constant c_1, c_2 and c_3 such that*

$$\begin{aligned}
&M\int_0^t E(s)ds + c_1||\operatorname{curl} H||^2\\
&\quad + \left(\frac{k}{2} - \frac{L^2 a_0^2||\alpha||_\infty}{a^2\varepsilon_1}\right)\int_0^t\int_\Omega \alpha(x)|u_t|^2 dx\, ds\\
&\quad + G_k(t) + \left(\frac{k\mu_0}{\gamma} - \frac{2\mu_0^2 L a_0^2}{a^2\varepsilon_1}\right)\int_0^t\int_\Omega |\operatorname{curl} h|^2 dx\, ds\\
&\le G_k(0) + c_2||\varphi||^2 + c_3||u_0||^2
\end{aligned} \tag{3.20}$$

where k and M and H are as in Lemma 3.3.

Proof: Due to our assumptions on the initial data h_0, the Hodge decomposition $h_0 = \nabla\Phi + \operatorname{curl}\varphi$ holds (see [4] page 232).

Then, we have the identity

$$\begin{aligned}
\int_\Omega h_0\cdot H\, dx &= -\int_\Omega \Phi \operatorname{div} H dx - \int_\Omega \varphi\cdot\operatorname{curl} H dx\\
&= -\int_\Omega \varphi\cdot\operatorname{curl} H\, dx
\end{aligned}$$

because div $H = 0$ in Ω. We can estimate for any $\delta > 0$

$$\Big|\int_\Omega h_0 \cdot H\, dx\Big| \leq \delta||\varphi||^2 + \frac{1}{4\delta}||\,\mathrm{curl}\, H||^2.$$

Thus

$$\begin{aligned}&\varepsilon\mu_0 \int_\Omega [h_0 - \mathrm{curl}(u_0 \times \widetilde{H}) \cdot H\, dx \\ &\quad \leq \varepsilon\mu_0\delta||\varphi||^2 + \frac{\varepsilon\mu_0}{4\delta}||\,\mathrm{curl}\, H||^2 - \varepsilon\mu_0 \int_\Omega \mathrm{curl}(u_0 \times \widetilde{H}) \cdot H\, dx.\end{aligned}$$

Since $|\widetilde{H}| = 1$ we deduce for any $\delta > 0$

$$\begin{aligned}\Big|\int_\Omega \mathrm{curl}(u_0 \times \widetilde{H}) \cdot H\, dx\Big| &= \Big|\int_\Omega (u_0 \times \widetilde{H}) \cdot \mathrm{curl}\, H\, dx\Big| \\ &\leq \int_\Omega |u_0||\,\mathrm{curl}\, H|dx \\ &\leq 2\delta||u_0||^2 + \frac{1}{8\delta}||\,\mathrm{curl}\, H||^2.\end{aligned}$$

Consequently, the last term on the right hand side of (3.9) can be bounded by

$$\varepsilon\mu_0\delta||\varphi||^2 + 2\varepsilon\mu_0\delta||u_0||^2 + \frac{3\varepsilon\mu_0}{8\delta}||\,\mathrm{curl}\, H||^2.$$

Choosing $\delta > \frac{3\gamma}{4}$ we obtain (3.20) with $c_1 = \varepsilon\mu_0(\frac{1}{2\gamma} - \frac{3}{8\delta}) > 0$, $c_2 = \varepsilon\mu_0\delta$ and $c_3 = 2\varepsilon\mu_0\delta$.

Lemma 3.5. *Let $0 < a^2 < b^2 < 4a^2$ and $\{u, h\}$ be the solution of system (1.1)–(1.3). Assuming hypotheses (H1) and (H2) we have*

$$G_k(t) \geq 0 \quad \textit{for all} \quad t \geq 0$$

and $k >> 1$ sufficiently large, where G_k is given as in Lemma 3.2.

Proof: First, we estimate $-\varepsilon(u_t, u)$ in $G_k(t)$. Using (H2) and Poincaré's lemma we have for any $\delta > 0$

$$\begin{aligned}-\varepsilon(u_t, u) &\leq \frac{\varepsilon}{2\delta}||u_t||^2 + \frac{\varepsilon\delta}{2}||u||^2 \\ &\leq \frac{\varepsilon}{\delta}E(t) + \frac{\varepsilon\delta}{2}\Big\{\int_{|x|\geq L} \frac{\alpha(x)}{a_0}|u|^2dx + \int_{\{x,|x|<L\}\cap\mathcal{O}} |u|^2dx\Big\} \\ &\leq \frac{\varepsilon}{\delta}E(t) + \frac{\varepsilon\delta}{2a_0}\int_\Omega \alpha(x)|u|^2dx + \frac{\delta\varepsilon}{2}C_L\int_\Omega |\nabla u|^2dx \\ &\leq \Big(\frac{\varepsilon}{\delta} + \frac{\varepsilon\delta}{2}C_L\Big)E(t) + \frac{\varepsilon\delta}{2a_0}\int_\Omega \alpha(x)|u|^2dx.\end{aligned} \tag{3.21}$$

Next, we estimate

$$
\begin{aligned}
-\int_\Omega \psi u_t \cdot (x : \nabla u)dx &\le \int_\Omega |x||\psi||u_t||\nabla u|dx \\
&\le La_0 \int_\Omega |u_t||\nabla u|dx \le \frac{La_0}{a}\int_\Omega \Big[\frac{|u_t|^2}{2} + \frac{a^2}{2}|\nabla u|^2\Big]dx \\
&\le \frac{La_0}{a}E(t) \quad \text{ for any } t \ge 0.
\end{aligned}
\tag{3.22}
$$

Next we choose $\delta > 0$ small such that $\delta/a_0 < 1$ and afterwords $k >> 1$ large such that

$$
\frac{\varepsilon}{\delta} + \frac{\varepsilon\delta}{2}C_L + \frac{La_0}{a} < k.
$$

Hence, using (3.21) and (3.22) with our above choice of k and δ we obtain

$$
\begin{aligned}
&-\int_\Omega \psi u_t \cdot (x : \nabla u)dx - \varepsilon \int_\Omega u_t \cdot u dx \\
&\le \frac{\varepsilon}{2}\int_\Omega \alpha |u|^2 dx + kE(t) \quad \text{ for any } t \ge 0
\end{aligned}
$$

which proves that $G_k(t) \ge 0$ for all $t \ge 0$.

Theorem 3.1. (Energy decay). *Let $0 < a^2 < b^2 < 4a^2$ and assume (H1)–(H2). Let $(u_0, u_1, h_0) \in H$ with $h_0 \in W \cap L^2(\Omega)$. Then, the total energy $E(t)$ (see (1.4)) associated with problem (1.1)–(1.3) satisfies*

$$
E(t) \le C(1+t)^{-1}
$$

where C is a positive constant which depends on the L^2-norms of the initial data $u_0, \nabla u_0, u_1$ and h_0.

Proof: Using Lemmas 3.4 and 3.5 we can write the inequality

$$
\begin{aligned}
&M\int_0^t E(s)ds + B_k \int_0^t ||\operatorname{curl} h||^2 ds \\
&\quad + A_k \int_0^t \int_\Omega \alpha(x)|u_t|^2 dx\, ds + c_1 ||\operatorname{curl} H||^2 \\
&\le G_k(0) + c_2||\varphi||^2 + c_3||u_0||^2.
\end{aligned}
$$

In particular

$$
\int_0^t E(s)ds \le C(u_0, u_1, \nabla u_0, h_0) \tag{3.23}
$$

where the constant can be estimated explicitely. Furthermore, since $E(t)$ is nonincreasing (by (1.5)) then for any $t > 0$ the inequality

$$
(1+t)E(t) \le E(0) + \int_0^t E(s)ds \tag{3.24}
$$

holds. Thus

$$(1+t)E(t) \leq E(0) + \int_0^t E(s)ds \leq E(0) + C.$$

Therefore the conclusion of Theorem 3.1 follows where C is a positive constant which depends only on the initial data.

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