Controllability of the 3D compressible Euler system

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Plan of the talk

1. Preliminaries on the 3D compressible Euler system
2. Main results
3. Sketch of the proof

Preliminaries on the 3D compressible Euler system

Consider the 3D compressible Euler system

$$\begin{aligned} \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) + \nabla p(\rho) &= \rho \boldsymbol{f}, \\ \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) &= 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \rho(0) &= \rho_0, \end{aligned}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\rho > 0$ are unknown velocity field and density of the gas, p is the pressure and $\boldsymbol{f}$ is the external force, $\boldsymbol{u}_0$ and ρ_0 are the initial conditions, $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

We have the following local well-posedness of the compressible Euler system.

Theorem

Let $(\boldsymbol{u}_0, \rho_0) \in \boldsymbol{H}^k \times H^k$, $k > n/2 + 1$ *and* $\boldsymbol{f} \in L^2([0,T), \boldsymbol{H}^k)$. *Then there exists* $T_0 > 0$, *which depends on*

$$\|\boldsymbol{u}_0\|_k + \|\rho_0\|_k + \|\boldsymbol{f}\|_{L^2(J_T, \boldsymbol{H}^k \times \boldsymbol{H}^k)},$$

such that the system has a unique solution $\boldsymbol{u} \in C([0, T_0), \boldsymbol{H}^k) \times C([0, T_0), H^k)$.

Let

$$\mathcal{R} : D(\mathcal{R}) \subset \boldsymbol{H}^k \times H^k \times L^2([0,T], \boldsymbol{H}^k) \to C([0,T], \boldsymbol{H}^k) \times C([0,T], H^k)$$
$$(\boldsymbol{u}_0, \rho_0, \boldsymbol{f}) \to (\boldsymbol{u}, \rho)$$

be the resolving operator of the system. Then the following assertions hold.

(i) $D(\mathcal{R})$ is open set.

(ii) The operator $\mathcal{R}$ is continuous.

(iii) The operator $\mathcal{R}$ is Lipschitz continuous from $\boldsymbol{H}^{k-1} \times H^{k-1} \times L^2([0,T], \boldsymbol{H}^{k-1})$ to $C([0,T], \boldsymbol{H}^{k-1}) \times C([0,T], H^{k-1})$.

We have the following blow-up criterion for the compressible Euler system.

Proposition

Let $(\boldsymbol{u}, \rho) \in C([0,T), \boldsymbol{H}^k) \times C([0,T), H^k)$ be a solution of Euler system. If for some $r > 0$

$$H := \sup_{t\in[0,T)} \|(\boldsymbol{u}, \rho)(t)\|_{C^{r+1}} < \infty,$$

then there exists $T_1(H) > T$ such that $(\boldsymbol{u}, \rho)$ extends to a solution defined on $[0, T_1)$.

Main results

Let us consider the controlled system

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) + \nabla p(\rho) = \rho(\boldsymbol{f} + \boldsymbol{\eta}), \tag{1}$$

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \rho(0) = \rho_0. \tag{3}$$

Definition

System (1), (2) with $\boldsymbol{\eta} \in \boldsymbol{X}$ is said to be controllable at time $T > 0$ if for any constants $\varepsilon > 0$, for any finite dimensional space $\boldsymbol{F} \subset \boldsymbol{H}^k \times H^k$ and for any functions $(\boldsymbol{u}_0, \rho_0), (\boldsymbol{u}_1, \rho_1) \in \boldsymbol{H}^k \times H^k$ satisfying $\int \rho_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int \rho_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ there is a control $\boldsymbol{\eta} \in \boldsymbol{X}$ such that

$$\|\mathcal{R}_T(\boldsymbol{u}_0, \rho_0, \boldsymbol{\eta}) - (\boldsymbol{u}_1, \rho_1)\|_{\boldsymbol{H}^k \times H^k} < \varepsilon,$$

$$\boldsymbol{P}_F(\mathcal{R}_T(\boldsymbol{u}_0, \rho_0, \boldsymbol{\eta})) = \boldsymbol{P}_F(\boldsymbol{u}_1, \rho_1).$$

For any finite-dimensional subspace $\boldsymbol{E} \subset \boldsymbol{H}^k$, we denote by $\mathcal{F}(\boldsymbol{E})$ the largest vector space $\boldsymbol{F} \subset \boldsymbol{H}^k$ such that for any $\boldsymbol{\eta}_1 \in \boldsymbol{F}$ there are vectors $\boldsymbol{\eta}, \zeta^1, \dots, \zeta^n \in \boldsymbol{E}$ satisfying the relation

$$\boldsymbol{\eta}_1 = \boldsymbol{\eta} - \sum_{i=1}^{n} (\zeta^i \cdot \nabla)\zeta^i. \tag{4}$$

It follows from $dim\mathcal{F}(E) < \infty$ and from the fact that if G_1 and G_2 satisfy (4), then so does $G_1 + G_2$ that $\mathcal{F}(E)$ is well defined.
We define $\boldsymbol{E_n}$ by the rule

$$\boldsymbol{E}_0 = \boldsymbol{E}, \quad \boldsymbol{E}_n = \mathcal{F}(\boldsymbol{E}_{n-1}) \quad \text{for} \quad n \geq 1, \quad \boldsymbol{E}_\infty = \bigcup_{n=1}^{\infty} \boldsymbol{E}_n.$$

Theorem

If $\boldsymbol{E} \subset \boldsymbol{H}^k$ *is a finite-dimensional subspace such that* $\boldsymbol{E}_\infty$ *is dense in* $\boldsymbol{H}^k$*, then system (1), (2) with* $\boldsymbol{\eta} \in C^\infty([0, T], \boldsymbol{E})$ *is controllable at time* $T > 0$*.*

Theorem

If $\boldsymbol{E} \subset \boldsymbol{H}^k$ is a finite-dimensional subspace such that $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^k$, then system (1), (2) with $\boldsymbol{\eta} \in C^\infty([0,T], \boldsymbol{E})$ is controllable at time $T > 0$.

Example

Let us introduce the functions

$$c^i_{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{e}_i \cos\langle \boldsymbol{m}, \boldsymbol{x}\rangle,\ s^i_{\boldsymbol{m}}(\boldsymbol{x}) = \boldsymbol{e}_i \sin\langle \boldsymbol{m}, \boldsymbol{x}\rangle, \quad i = 1, 2, 3,$$

where $\boldsymbol{m} \in \mathbb{Z}^3$ and $\{\boldsymbol{e}_i\}$ is the standard basis in $\mathbb{R}^3$.
If

$$\boldsymbol{E} = span\{\boldsymbol{c}^{\boldsymbol{i}}_{\boldsymbol{m}}, \boldsymbol{s}^{\boldsymbol{i}}_{\boldsymbol{m}},\ |m| \le 3\},$$

then $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^k$.

Let us denote

$$A(\boldsymbol{u}_0, \rho_0, \boldsymbol{X}) = \{\mathcal{R}_T(\boldsymbol{u}_0, \rho_0, \boldsymbol{\eta}), \ \ \boldsymbol{\eta} \in \boldsymbol{X}\} \subset \boldsymbol{H}^k \times H^k.$$

Theorem

Let $k \geq 4$, $(\boldsymbol{u}_0, \rho_0) \in \boldsymbol{H}^k \times H^k$, and $\boldsymbol{E} \subset \boldsymbol{H}^k$ be any finite-dimensional subspace. Then $A(\boldsymbol{u}_0, \rho, L^1([0, \infty), \boldsymbol{E}))$ does not contain a ball of the space $\boldsymbol{H}^k \times H^k$.

Li and Rao, Glass

Coron, Fursikov, Imanuvilov

Agrachev and Sarychev

Shirikyan

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Sketch of the proof

Reduction to $(\varepsilon, \boldsymbol{K})$-controllability

Definition

Problem is said to be $(\varepsilon, \boldsymbol{K})$-controllable with $\boldsymbol{\eta} \in \boldsymbol{X}$ if there is a continuous mapping $\boldsymbol{\Psi} : \boldsymbol{K} \to \boldsymbol{X}$ such that

$$\sup_{(\hat{\boldsymbol{u}},\hat{\rho})\in \boldsymbol{K}} \|\mathcal{R}_T(\boldsymbol{u}_0, \rho_0, \boldsymbol{\Psi}(\hat{\boldsymbol{u}}, \hat{\rho})) - (\hat{\boldsymbol{u}}, \hat{\rho})\|_{\boldsymbol{H}^k \times H^k} < \varepsilon,$$

Reduction to (ε, K)-controllability

Definition

Problem is said to be (ε, K)-controllable with $\eta \in X$ if there is a continuous mapping $\Psi : K \to X$ such that

$$\sup_{(\hat{u},\hat{\rho})\in K} \|\mathcal{R}_T(u_0, \rho_0, \Psi(\hat{u}, \hat{\rho})) - (\hat{u}, \hat{\rho})\|_{H^k \times H^k} < \varepsilon,$$

Theorem

If for any constant $\varepsilon > 0$ and compact set $K \subset H^k \times H^k$ system (1), (2) is (ε, K)-controllable, then it is controllable.

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$$\partial_t u + B(u + \zeta) = \eta. \tag{6}$$

(i) Equation (5) is $(\varepsilon, \boldsymbol{K})$-controllable with $\boldsymbol{E}_N$-valued controls for some $N \geq 1$.

(ii) $(\varepsilon, \boldsymbol{K})$-controllability of (5) with $\eta \in \boldsymbol{E}_n$ is equivalent to $(\varepsilon, \boldsymbol{K})$-controllability of (6) with $\eta, \zeta \in \boldsymbol{E}_n$.

(iii) $(\varepsilon, \boldsymbol{K})$-controllability of (5) with $\eta \in \boldsymbol{E}_{n+1}$ is equivalent to $(\varepsilon, \boldsymbol{K})$-controllability of (6) with $\eta, \zeta \in \boldsymbol{E}_n$.

Example

Let us consider

$$\dot{u} = \eta(t), \quad u(0) = u_0, \quad \eta(t), u(t) \in \mathbb{R}.$$

Then

$$A_T(u_0, [a, b]) = A_T(u_0, \{a, b\}).$$

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Example

Let $A, B, C \in \mathbb{R}^2$

$$\dot{u} = \eta(t), \quad u(0) = u_0, \quad \eta(t), u(t) \in \mathbb{R}^2.$$

Then

$$A_T(u_0, \Delta_{ABC}) = A_T(u_0, \{A, B, C\}).$$

We need to consider the control system

$$\rho(\partial_t \boldsymbol{u} + ((\boldsymbol{u} + \zeta) \cdot \nabla)(\boldsymbol{u} + \zeta)) + \nabla p(\rho) = \rho(\boldsymbol{f} + \boldsymbol{\eta}), \qquad (7)$$

$$\partial_t \rho + \nabla \cdot (\rho(\boldsymbol{u} + \zeta)) = 0. \qquad (8)$$

For any $(\boldsymbol{u}_0, \rho_0)$ and $(\boldsymbol{u}_1, \rho_1)$ we find controls $\zeta, \boldsymbol{\eta}$ such that the solution of (7)-(8) links $(\boldsymbol{u}_0, \rho_0)$ and $(\boldsymbol{u}_1, \rho_1)$. Combination of a perturbative result and of the fact that $\boldsymbol{E}_\infty$ is dense in $\boldsymbol{H}^k$ implies $(\varepsilon, \boldsymbol{K})$-controllability of (7)-(8) with $\boldsymbol{E}_N$-valued controls, $N \gg 1$.

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To show (ii), without loss of generality, we can assume that $\zeta(0) = \zeta(T) = 0$. If $(\boldsymbol{u}, \rho)$ is the solution of (7)-(8), then

$$(\boldsymbol{u}+\zeta, \rho) = \mathcal{R}(\boldsymbol{u}_0, \rho_0, \boldsymbol{\eta} - \partial_t\zeta).$$

Thus, we have (i) and (ii).

Suppose $\boldsymbol{\eta}_1 \in \boldsymbol{E}_{n+1} = \mathcal{F}(\boldsymbol{E}_n)$ and $(\boldsymbol{u}_1, \rho_1) := \mathcal{R}(\boldsymbol{u}_0, \rho_0, \boldsymbol{\eta}_1)$. Then there are vectors $\zeta^1, \dots, \zeta^p, \boldsymbol{\eta} \in \boldsymbol{E}_n$ and positive constants $\lambda_p, \dots, \lambda_n$ whose sum is equal to 1 such that

$$(\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1 - \boldsymbol{\eta}_1 = \sum_{j=1}^{p} \lambda_j((\boldsymbol{u}_1 + \zeta^j) \cdot \nabla)(\boldsymbol{u}_1 + \zeta^j) - \boldsymbol{\eta}.$$

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$$(\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_1 - \boldsymbol{\eta}_1 = \sum_{j=1}^{p} \lambda_j((\boldsymbol{u}_1 + \zeta^j) \cdot \nabla)(\boldsymbol{u}_1 + \zeta^j) - \boldsymbol{\eta}.$$

Let $\zeta(t)$ be a 1-periodic function such that

$$\zeta(t) = \zeta^j \text{ for } 0 \leq t - (\lambda_1 + \ldots + \lambda_{j-1}) < \lambda_j, \quad j = 1, \ldots, p,$$

and let $\zeta_k(t) = \zeta(\frac{kt}{T})$. Then

$$\rho(\partial_t \boldsymbol{u}_1 + ((\boldsymbol{u}_1 + \zeta_k) \cdot \nabla)(\boldsymbol{u}_1 + \zeta_k)) + \nabla p(\rho_1) = \rho(\boldsymbol{f} + \boldsymbol{\eta} + \boldsymbol{f}_k(t)),$$

where $\|\boldsymbol{f}_k\| \to 0$.

We show that the controllability of compressible Euler system with $\boldsymbol{\eta} \in \boldsymbol{E}_{n+1}$ is equivalent to that of the system

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) + \nabla p(\rho) = \rho(\boldsymbol{f} + \boldsymbol{\eta}), \tag{9}$$

$$\partial_t \rho + \nabla \cdot (\rho(\boldsymbol{u} + \zeta)) = 0. \tag{10}$$

with $\zeta, \boldsymbol{\eta} \in \boldsymbol{E}_n$.

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with $\zeta, \boldsymbol{\eta} \in \boldsymbol{E}_n$.
If ζ_k is a sequence of a smooth functions such that

$$\int_0^t \zeta_k(s, x) \mathrm{d}s \to 0 \text{ as } k \to \infty,$$

then for large k the solution of (10) is close to that of the equation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

Definition

For any $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimal number of sets of diameters not exceeding 2ε that are needed to cover K. The Kolmogorov ε-entropy of K is defined as $H_\varepsilon(K) = \ln N_\varepsilon(K)$.

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Example

Let Q is a ball in H^k. Then

$$H_\varepsilon(Q, H^s) \sim \left(\frac{1}{\varepsilon}\right)^{\frac{n}{k-s}}.$$

Suppose that there is a ball $B \in L^1([0,T],E)$ such that the set $A(\boldsymbol{u}_0,\rho_0,B)$ contains a ball $Q \in \boldsymbol{H}^k \times H^k$. Since $Q \subset A(\boldsymbol{u}_0,\rho_0,B)$, we have

$$H_\varepsilon(A(\boldsymbol{u}_0,\rho_0,B),\boldsymbol{H}^{k-1}\times H^{k-1}) \geq H_\varepsilon(Q,\boldsymbol{H}^{k-1}\times H^{k-1}) \geq C\frac{1}{\varepsilon^3}.$$

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On the other hand, the Lipschitz-continuity of $\mathcal{R}$ implies

$$H_\varepsilon(A(\boldsymbol{u}_0,\rho_0,B),\boldsymbol{H}^{k-1}\times H^{k-1}) \leq CH_\varepsilon(B,\boldsymbol{H}^{k-1}) \leq C\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}.$$

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