Stabilization of Euler system in an infinite strip
Let

\[ D := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}. \]

Let us take two open intervals \((a, b), (a + d, b + d) \subset \mathbb{R}\) and denote

\[ \Gamma_0 = (a, b) \times \{1\} \cup (a + d, b + d) \times \{-1\}. \]

We consider

\[
\begin{align*}
\dot{u} + \langle u, \nabla \rangle u + \nabla p &= 0 \quad \text{in} \quad D, \\
\text{div} \ u &= 0 \quad \text{in} \quad D, \\
\mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0, \\
\mathbf{u}(x, 0) &= \mathbf{u}_0(x).
\end{align*}
\]
For any integer $s > 0$ we define $H^s(D)$ as the space of distributions $u$ in $D$ with $\nabla u \in H^{s-1}(D)$. We equip $H^s(D)$ with the semi-norm

$$\|u\|_{H^s(D)} := \|\nabla u\|_{s-1}.$$ 

We denote by $\dot{H}^s(D)$ the quotient space $H^s(D)/\mathbb{R}$. 
For any initial data $u_0 \in H^s(D)$, $s \geq 4$ such that

$$\text{div } u_0 = 0,$$
$$u_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0,$$
$$\| \exp(\alpha \langle x_1 \rangle^{2+\beta}) \text{curl } u_0(x_1, x_2) \|_{s-1} < \infty$$

and for any constant $c \in \mathbb{R}$ there is a solution $(u, p) \in C(\mathbb{R}_+, C(\bar{D}) \cap \dot{H}^s(D)) \times C(\mathbb{R}_+, \dot{H}^{s+1}(D))$ with

$$\lim_{t \to \infty} (\| u(\cdot, t) - (c, 0) \|_{L^\infty} + \| \nabla u(\cdot, t) \|_{s-1} + \| \nabla p(\cdot, t) \|_{s-1}) = 0.$$
Let us take $M > 0$ such that

$$\|\frac{u_0}{M}\|_s < \varepsilon, \quad \left|\frac{c}{M}\right| < \varepsilon.$$  

Then there exists a solution $(u_M, p_M)$ of Euler system with initial condition $u_M(0) = \frac{u_0}{M}$, such that

$$\lim_{t \to \infty} \|u_M(\cdot, t) - \left(\frac{c}{M}, 0\right)\|_{L^\infty} = 0.$$  

Then $(u, p) = (Mu_M(x, Mt), M^2 p_M(x, Mt))$ is a solution of our stabilization problem.
Return Method (Coron)

One looks for a particular solution $(\overline{u}, \overline{p})$ such that the linearized control system around $(\overline{u}, \overline{p})$ is controllable.
Let $\phi^f : D \times [0, \infty) \rightarrow \mathbb{R}^2$ is the flow associated to $f$:
\[
\frac{\partial \phi^f}{\partial t} = f(\phi^f, t), \quad \phi^f(x, 0) = x.
\]
Let $\phi^f : D \times [0, \infty) \to \mathbb{R}^2$ is the flow associated to $f$:
\[
\frac{\partial \phi^f}{\partial t} = f(\phi^f, t), \quad \phi^f(x, 0) = x.
\]

**Proposition**

For any $c \in \mathbb{R}$ there are open balls $B^i \subset \mathbb{R}^2$, times $t_i$ and a solution $(\bar{u}, \bar{p})$ such that

\[
\overline{D} \subset \bigcup_{i=1}^{\infty} B^i,
\]
\[
\phi^i(\bar{u}, t_i) \notin D,
\]
\[
\lim_{t \to \infty} \| \bar{u}(\cdot, t) - (c, 0) \|_{L^\infty} = 0.
\]
Let $\tilde{u}$ be sufficiently close to $\bar{u}$ and $w^l \in C([0, \infty), H^s(D))$ be the solution of

$$
\dot{w}^l + \langle \tilde{u}, \nabla \rangle w^l = 0 \text{ in } D \times [0, \infty),
$$

$$
w^l(0) = \kappa^l \text{ curl}(u_0).
$$
Let \( \tilde{u} \) be sufficiently close to \( u \) and \( w^l \in C([0, \infty), H^s(D)) \) be the solution of

\[
\dot{w}^l + \langle \tilde{u}, \nabla \rangle w^l = 0 \text{ in } D \times [0, \infty),
\]

\[
w^l(0) = \kappa^l \text{ curl}(u_0).
\]

**Proposition**

There is a constant \( \nu > 0 \) such that the function \( \tilde{u} \) can be chosen in a way that, for any \( \tilde{u} \) satisfying the inequality

\[
\int_0^\infty \| \tilde{u}(t) - \bar{u}(t) \|_s dt \leq \nu,
\]

we have \( \phi(\tilde{u}(B^i, t_i), D) \notin D \) for any \( i \geq 1 \).
Then $\phi \tilde{u}(B^l, t_l) \notin D$ for any $l$. Thus, we have

$$w^l(\cdot, t_l)|_D = 0.$$
Then $\phi(B^l, t_l) \notin D$ for any $l$. Thus, we have

$$w^l(\cdot, t_l)|_D = 0.$$  

We define,

$$w(\cdot, t) = \sum_{l=i+1}^{\infty} w^l(\cdot, t) \quad \text{for } t \in [t_i, t_{i+1}].$$
Then \( \phi \tilde{u}(B^l, t_I) \notin D \) for any \( l \). Thus, we have

\[
w^l(\cdot, t_I)|_D = 0.
\]

We define,

\[
w(\cdot, t) = \sum_{i=i+1}^{\infty} w^l(\cdot, t) \quad \text{for } t \in [t_i, t_{i+1}].
\]

Then \( w \) is a solution of

\[
\dot{w} + \langle \tilde{u}, \nabla \rangle w = 0 \quad \text{in } D \times [0, \infty),
\]

\[
w(0) = \text{curl } u_0 \quad \text{in } D.
\]
We take the solution of the problem

\[
\text{curl } \mathbf{v} = \mathbf{w}, \quad \text{div } \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}_{\Gamma \setminus \Gamma_0} = 0.
\]

Now we define mapping \( F(\tilde{u}) := \mathbf{v} \) and we show that \( F \) admits a fixed point, which is the solution of our stabilization problem.
Construction of $\overline{u}$.
We seek a particular solution in the form $\overline{u} = \nabla \theta(x, t)$, where

$$\Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0.$$
Construction of $\overline{u}$.
We seek a particular solution in the form $\overline{u} = \nabla \theta(x, t)$, where

$$\Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0.$$

We show that

$$\mathbb{R}^2 = \{ \nabla \theta(x_0) : \Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0 \text{ and } \| x_1^2 \partial_i \partial_j \theta \|_s < \infty \}. $$
Construction of $\overline{u}$.
We seek a particular solution in the form $\overline{u} = \nabla \theta(x, t)$, where
\[
\Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0.
\]

We show that
\[
\mathbb{R}^2 = \{ \nabla \theta(x_0) : \Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0 \text{ and } \|x_1^2 \partial_i \partial_j \theta\|_s < \infty \}.
\]

Then there exist functions $\theta_i$ and a covering $\{B_i\}_{i=1}^N$ of $[0, 1] \times [-1, 1]$ such that
\[
\phi^{\nabla \theta_i + (c, 0)}(B_i, 1) \notin \overline{D}.
\]
We denote $B^{2kN+j} := B(x_j, r_j) + (k, 0)$ $j = 1, \ldots, N$. Let

$$
\theta^{2kN+j}(x, t) = \begin{cases} 
(-k - c)x_1 h'(t) & \text{for } t \in [0, 1], \\
\theta^j(x, t - 1) & \text{for } t \in [1, 2],
\end{cases}
$$

where $h \in C^\infty([0, 1])$ is

$$
\begin{align*}
h(t) &= 0 \quad \text{for } t \in [0, 1/4], \\
h(t) &= 1 \quad \text{for } t \in [3/4, 1], \\
|h(t)| &\leq 1 \quad \text{for } t \in [0, 1].
\end{align*}
$$
\[ \tilde{\theta}^i(x, t) := \frac{\theta^i(x, \frac{t}{\tau_i})}{\tau_i}, \quad \tau_i := i \sup_{t \in [0,2]} \| \nabla \theta^i(\cdot, t) \|_s, \quad t_i := \sum_{j=1}^{i} \tau_j. \]

Then we define

\[ \overline{u}(x, t) := \nabla \tilde{\theta}^i(x, t - t_{i-1}) + (c, 0) \quad \text{pour } t \in [t_{i-1}, t_i]. \]
Thank you for your attention