Stabilization of Euler system in an infinite strip

Let

$$D := \{(x_1, x_2) :\ x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}.$$

Let us take two open intervals $(a, b), (a + d, b + d) \subset \mathbb{R}$ and denote

$$\Gamma_0 = (a, b) \times \{1\} \cup (a + d, b + d) \times \{-1\}.$$

We consider

$$\begin{aligned}
\dot{\boldsymbol{u}} + \langle \boldsymbol{u}, \nabla \rangle \boldsymbol{u} + \nabla p = 0 \qquad & \text{in} \quad D, \\
\operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in} \quad D, \\
\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad & \text{on} \quad \Gamma \setminus \Gamma_0, \\
\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x). &
\end{aligned}$$

For any integer $s > 0$ we define $\mathcal{H}^s(D)$ as the space of distributions u in D with $\nabla u \in H^{s-1}(D)$. We equip $\mathcal{H}^s(D)$ with the semi-norm

$$\|u\|_{\mathcal{H}^s(D)} := \|\nabla u\|_{s-1}.$$

We denote by $\dot{H}^s(D)$ the quotient space $\mathcal{H}^s(D)/\mathbb{R}$.

Theorem

For any initial data $\boldsymbol{u}_0 \in H^s(D)$, $s \geq 4$ *such that*

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{u}_0 = 0, \\ &\boldsymbol{u}_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \\ &\| \exp(\alpha \langle x_1 \rangle^{2+\beta}) \,\mathrm{curl}\, \boldsymbol{u}_0(x_1, x_2) \|_{s-1} < \infty \end{aligned}$$

and for any constant $c \in \mathbb{R}$ *there is a solution* $(\boldsymbol{u}, p) \in C(\mathbb{R}_+, C(\overline{D}) \cap \dot{\boldsymbol{H}}^s(D)) \times C(\mathbb{R}_+, \dot{H}^{s+1}(D))$ *with*

$$\lim_{t \to \infty} (\|\boldsymbol{u}(\cdot, t) - (c, 0)\|_{L^\infty} + \|\nabla \boldsymbol{u}(\cdot, t)\|_{s-1} + \|\nabla p(\cdot, t)\|_{s-1}) = 0.$$

Let us take $M > 0$ such that

$$\|\frac{u_0}{M}\|_s < \varepsilon, \quad |\frac{c}{M}| < \varepsilon.$$

Then there exists a solution (u_M, p_M) of Euler system with initial condition $u_M(0) = \frac{u_0}{M}$, such that

$$\lim_{t\to\infty} \|u_M(\cdot, t) - (\frac{c}{M}, 0)\|_{L^\infty} = 0.$$

Then $(u, p) = (Mu_M(x, Mt), M^2 p_M(x, Mt))$ is a solution of our stabilization problem.

Return Method (Coron)

One looks for a particular solution $(\overline{\boldsymbol{u}}, \overline{p})$ such that the linearized control system around $(\overline{\boldsymbol{u}}, \overline{p})$ is controllable.

Let $\phi^f : D \times [0, \infty) \to \mathbb{R}^2$ is the flow associated to f:

$$\frac{\partial \phi^f}{\partial t} = f(\phi^f, t), \quad \phi^f(x, 0) = x.$$

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Proposition

For any $c \in \mathbb{R}$ there are open balls $B^i \subset \mathbb{R}^2$, times t_i and a solution $(\overline{\boldsymbol{u}}, \overline{p})$ such that

$$\overline{D} \subset \cup_{i=1}^{\infty} B^i,$$

$$\phi^{\overline{\boldsymbol{u}}}(B^i, t_i) \notin D,$$

$$\lim_{t \to \infty} \|\overline{\boldsymbol{u}}(\cdot, t) - (c, 0)\|_{L^\infty} = 0.$$

Let $\tilde{\boldsymbol{u}}$ be sufficiently close to $\overline{\boldsymbol{u}}$ and $w^l \in C([0,\infty), H^s(D))$ be the solution of

$$\dot{w}^l + \langle \tilde{\boldsymbol{u}}, \nabla \rangle w^l = 0 \text{ in } D \times [0,\infty),$$
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Proposition

There is a constant $\nu > 0$ such that the function $\overline{\boldsymbol{u}}$ can be chosen in a way that, for any $\tilde{\boldsymbol{u}}$ satisfying the inequality

$$\int_0^\infty \|\tilde{\boldsymbol{u}}(t) - \overline{\boldsymbol{u}}(t)\|_s \mathrm{d}t \leq \nu,$$

we have $\phi^{\tilde{\boldsymbol{u}}}(B^i, t_i) \notin \overline{D}$ for any $i \geq 1$.

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$$w(\cdot, t) = \sum_{l=i+1}^{\infty} w^l(\cdot, t) \quad \text{for } t \in [t_i, t_{i+1}].$$

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Then w is a solution of

$$\begin{aligned} \dot{w} + \langle \tilde{\boldsymbol{u}}, \nabla \rangle w = 0 \qquad & \text{in} \quad D \times [0, \infty), \\ w(0) = \operatorname{curl} u_0 \qquad & \text{in} \quad D. \end{aligned}$$

We take the solution of the problem

$$\operatorname{curl} \boldsymbol{v} = w, \qquad \operatorname{div} \boldsymbol{v} = 0, \qquad \boldsymbol{v} \cdot \boldsymbol{n}_{\Gamma \setminus \Gamma_0} = 0.$$

Now we define mapping $F(\tilde{\boldsymbol{u}}) := \boldsymbol{v}$ and we show that F admits a fixed point, which is the solution of our stabilization problem.

Construction of $\overline{\boldsymbol{u}}$.
We seek a particular solution in the form $\overline{\boldsymbol{u}} = \nabla\theta(x,t)$, where

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We show that

$$\mathbb{R}^2 = \{\nabla\theta(x_0) : \Delta\theta = 0, \quad \partial_n\theta|_{\Gamma\setminus\Gamma_0} = 0 \text{ and } \|x_1^2\partial_i\partial_j\theta\|_s < \infty\}.$$

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Then there exist functions θ_i and a covering $\{B_i\}_{i=1}^N$ of $[0,1]\times[-1,1]$ such that

$$\phi^{\nabla\theta_i+(c,0)}(B_i,1) \notin \overline{D}.$$

We denote $B^{2kN+j} := B(x_j, r_j) + (k, 0)$ $j = 1, \ldots, N$. Let

$$\theta^{2kN+j}(x,t) = \begin{cases} (-k-c)x_1 h'(t) & \text{for } t \in [0,1], \\ \theta^j(x, t-1) & \text{for } t \in [1,2], \end{cases}$$

where $h \in C^\infty([0,1])$ is

$$\begin{aligned} h(t) &= 0 \quad && \text{for } t \in [0, 1/4], \\ h(t) &= 1 \quad && \text{for } t \in [3/4, 1], \\ |h(t)| &\leq 1 \quad && \text{for } t \in [0,1]. \end{aligned}$$

$$\tilde{\theta}^i(x,t) := \frac{\theta^i(x, \frac{t}{\tau_i})}{\tau_i}, \quad \tau_i := i \sup_{t\in[0,2]} \|\nabla\theta^i(\cdot,t)\|_s, \quad t_i := \sum_{j=1}^{i} \tau_j.$$

Then we define

$$\overline{\boldsymbol{u}}(x,t) := \nabla\tilde{\theta}^i(x, t - t_{i-1}) + (c, 0) \quad \text{pour } t \in [t_{i-1}, t_i].$$

Thank you for your attention