ATTRACTOR FOR LATTICES WITH NONLINEAR DAMPING [1]

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[1] joint work with J.C. Oliveira, J. Math. Anal. Appl. 370 (2010) 726-739

We consider lattices that have the form

$$\ddot{u}_n + (-1)^p \Delta^p u_n + \alpha u_n + h(u_n) + V(n)\, g(\dot{u}_n) = f_n, \tag{1}$$

where $u_n = u_n(t)$, $\dot{u}_n = \frac{du_n}{dt}$, $\ddot{u}_n = \frac{d^2 u_n}{dt^2}$, $\Delta u_n = u_{n+1} + u_{n-1} - 2u_n$, with $n \in \mathbb{Z}$; α is a positive constant, p is any positive integer, and h, V and g are real functions satisfying certain conditions.

introduction (cont.)

- Semi-discrete Klein-Gordon equation ($p = 1$):

$$\ddot{u}_n - \Delta u_n + \alpha u_n + h(u_n) + V(n)\, g(\dot{u}_n) = f_n,$$

- Semi-discrete beam equation (p=2):

$$\ddot{u}_n + \Delta^2 u_n + \alpha u_n + h(u_n) + V(n)\, g(\dot{u}_n) = f_n,$$

where $\Delta^2 u_n = u_{n+2} - 4u_{n+1} + 6u_n - 4u_{n-1} + u_{n-2}$.

Introduction (cont.)

We study the existence and uniqueness of solutions of the *IVP*[1]

$$\begin{aligned} &\ddot{u}_n(t) + (-1)^p \Delta^p u_n(t) + \alpha u_n(t) + h(u_n(t)) + V(n) g(\dot{u}_n(t)) = f_n \\ &u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}, \end{aligned} \tag{2}$$

and

Existence of global attractor for the semigroup $\{S(t)\}_{t \geq 0}$ associated with (2).

(1) We followed G.Perla Menzala and V.V. Konotop, Applicable Analysis 75 (2000) 157-173.

Existence of solutions

We consider the IVP (2) in the (energy) space: For $\tau > 0$ $H_\alpha(\tau)$ is the space of functions $u = (u_n(t))$ such that $u_n \in C^1([0,\tau]; \mathbb{R})$, $\forall n \in \mathbb{Z}$ and

$$\sup_{0 \le t \le \tau} \sum_{n=-\infty}^{+\infty} [(\dot{u}_n(t))^2 + (D^p u_n(t))^2 + \alpha\,(u_n(t))^2] < +\infty.$$

The space $H_\alpha(\tau)$ becomes a Banach space with norm given by

$$\|u\|_{\alpha,\tau} = \left\{ \sup_{0 \le t \le \tau} \sum_{n=-\infty}^{+\infty} [(\dot{u}_n(t))^2 + (D^p u_n(t))^2 + \alpha\,(u_n(t))^2] \right\}^{1/2}.$$

Remark:

$$D^p := \begin{cases} \Delta^{\frac{p}{2}}, & p \text{ even} \\ \partial^+ \Delta^{\frac{p-1}{2}}, & p \text{ odd} \end{cases},$$

where $\partial^+ u_n = u_{n+1} - u_n$.

Existence of solutions (cont.)

We assume that the functions h, V and g satisfy the following conditions (H):

$$h \in C^1(\mathbb{R};\mathbb{R}), h(0) = 0 \text{ and } \tilde{h}(s) = \int_0^s h(\sigma)\, d\sigma \geq 0, \forall s \in \mathbb{R}. \tag{3}$$

$$sh(s) \geq c_0 \tilde{h}(s), \forall s \in \mathbb{R}, \text{ for some positive constant } c_0. \tag{4}$$

$$g \in C^1(\mathbb{R};\mathbb{R}), g(0) = 0 \text{ and } g'(s) \geq c_1, \forall s \in \mathbb{R}, \text{for some positive constant } c_1 \tag{5}$$

$V(n)$ is a positive bounded function on $\mathbb{Z}$ such that $V(n) \geq V_0 > 0$, for all $|n| \geq n_0$.

Existence of solutions (cont.)

Theorem 1.1*: Assume conditions (H), except (4) and also that* $(u_{0,n})$, $(u_{1,n})$ *and* (f_n) *are in* ℓ^2*. Then the IVP (2) has a unique solution* $u = (u_n(t))$ *defined in* $[0, \infty)$*, such that* $u \in H_\alpha(\tau), \forall 0 < \tau < \infty$.

Remark: ℓ^2 is the space of all sequences of real numbers $(a_n)_{n \in \mathbb{Z}}$, such that $\sum_{n=-\infty}^{+\infty} a_n^2 < +\infty$.

Existence of solutions (cont.)

Step 1 We solve the linear problem using Green´s function:

$$\begin{cases} \ddot{v}_n + (-1)^P \Delta^p v_n + \alpha v_n = f_n \\ v_n(0) = u_{0,n},\ \dot{v}_n(0) = u_{1,n} \end{cases} \tag{6}$$

with $(u_{0,n})$, $(u_{1,n})$ and (f_n) in ℓ^2. We obtain

$$v_n(t) = \sum_{m=-\infty}^{+\infty} \left[\dot{G}(n-m,t)\, u_{0,m} + G(n-m,t)\, u_{1,m} \right] \tag{7}$$

where $G(k,t)$ (Green's function) is given by

$$G(k,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(wt)}{w} \cos kx \, dx, \tag{8}$$

where $w = w(x) = \left[\alpha + 4^p \sin^{2p}(x/2) \right]^{1/2}$.

Existence of solutions (cont.)

Remark: $G(k,t)$ solves the IVP

$$\begin{aligned} &\ddot{G}(k,t) + (-1)^p \Delta^p G(k,t) + \alpha G(k,t) = 0 \\ &G(k,0) = 0, \quad \dot{G}(k,0) = \delta_k, \end{aligned} \tag{9}$$

where $\delta_k = \begin{cases} 1, & k \neq 0 \\ 0, & k = 0 \end{cases}$.

and following estimates hold

$$|G(k,t)| \leq \frac{C(\alpha)}{k^2}(|t| + |t|^2), \quad |G(0,t)| \leq |t| \tag{10}$$

$$|\dot{G}(k,t)| \leq \frac{C(\alpha)}{k^2}(|t| + |t|^2), \quad |\dot{G}(0,t)| \leq 1, \tag{11}$$

Existence of solutions (cont.)

Step 2 We solve the integral equation associated with (2)

$$u_n(t) = v_n(t) + \sum_{m=-\infty}^{+\infty} \int_0^t G(n-m, t-s)[-F(u_m(s), \dot{u}_m(s)) + f_m(s)]\, ds, \quad (12)$$

where $G(n,t)$ is the Green´s function and $F(u_n, \dot{u}_n) = h(u_n) + V(n)\, g(\dot{u}_n)$.
For the proof we consider the closed subset of $H_\alpha(\tau)$:

$$Y_R(\tau) = \{u = (u_n(t))\,;\ u \in H_\alpha(\tau),\ \|u - v\|_{\alpha,\tau} \le R,\ u_n(0) = u_{o,n} \text{ and } \dot{u}_n(0) = u_{1,n},\ \forall n \in \mathbb{Z}\};$$

Existence of solutions (cont.)

For $u \in Y_R(\tau)$ we define the function Pu defined by $P(u)(t) = \left(\tilde{P}u_n(t)\right)$, where

$$\tilde{P}u_n(t) = v_n(t) + \sum_{m=-\infty}^{+\infty} \int_0^t G(n-m, t-s)\left[-F(u_m(s), \dot{u}_m(s)) + f_m(s)\right] ds. \quad (13)$$

and show that

$$(a)\ P:\ Y_R(\tau) \to Y_R(\tau) \quad \text{and} \quad (b)\ P \text{ is a contraction in } Y_R(\tau),$$

if τ is chosen sufficiently small. Next, we differentiate twice equation (12). This gives a local solution.

Existence of solutions (cont.)

Step 3 Using the energy equation we show that the local solution can be extend to interval$[0,\infty)$.
Multiplying equation (2) by $\dot{u}_n$ and summing in $\mathbb{Z}$ we obtain

$$\frac{d}{dt}E(t) = -\sum_{n=-\infty}^{+\infty} V(n)\dot{u}_n(t)g(\dot{u}_n(t)) \leq 0, \tag{14}$$

where

$$E(t) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\left[(\dot{u}_n(t))^2 + (D^p u_n(t))^2 + \alpha\,(u_n(t))^2\right] + \sum_{n=-\infty}^{+\infty}\tilde{h}(u_n(t)) - \sum_{n=-\infty}^{+\infty} f_n\, u_n(t \tag{15}$$

Using (14) and (15) we deduce that

$$||u||^2_{\alpha,\tau} \leq 4|E(0)| + \frac{16}{\alpha}||f||^2_{\ell^2} < +\infty,\ \forall 0 < \tau < \tau_{\max},$$

from which the result follows.

Global attractor

We consider as phase space the space $H = \ell^2 \times \ell^2$, with the usual norm given by

$$\|(a,b)\|_H = \left(\|a\|_{\ell^2}^2 + \|b\|_{\ell^2}^2\right)^{1/2},$$

$\forall a, b \in \ell^2$.
Let us denote by $\{S(t)\}_{t\geq 0}$ the semigroup associated to the IVP (2), which is defined by

$$S(t)(u_0, u_1) = (u(t), \dot{u}(t))$$

where $u = (u_n(t))$ is the global solution of (2) and $u_0 = (u_{0,n})$, $u_1 = (u_{1,n})$.

Global attractor(Main Result)

Theorem 2.1 *Assume hypotheses (H) and also that* $u_0 = (u_{0,n})$, $u_1 = (u_{1,n})$ *and* $f = (f_n)$ *belong to* ℓ^2. *Then the semigroup* $\{S(t)\}_{t\geq 0}$ *has a global attractor in* H.

In order to proof Theorem 2.1, according to Theorem 1 of Teman's book ([4]), it is sufficient to show that $\{S(t)\}_{t\geq 0}$ has an absorbing set in H and that it is asymptotically compact in H.

In ours proofs we use a difference inequality due to M. Nakao.

Global attractor: Some Definitions

Let $\{S(t)\}_{t\geq 0}$ be a semigroup of continuous operator in a Banach space H.

Definition 1. A set $B_0 \subset H$ is an **absorbing set** *for $\{S(t)\}_{t\geq 0}$ if for each bounded set B of H, there exists a time $\tau = \tau(B)$ such that $S(t)B \subset B_0$ for all $t \geq \tau$.*

Definition 2. $\{S(t)\}_{t\geq 0}$ is **asymptotically compact** *in H if for any bounded sequence $(u_m)_{m\in\mathbb{N}}$ in H and $t_m \to +\infty$, $\{S(t_m)u_m; m \in \mathbb{N}\}$ is relatively compact in H.*

Definition 3. $\mathcal{A} \subset H$ is a **global attractor** *for $\{S(t)\}_{t\geq 0}$ if $\mathcal{A}$ is compact, invariant and attracts all bounded sets of H,that is, $\forall B \subset H$, bounded, $d(S(t)B, \mathcal{A}) = \sup_{x\in B} \inf_{y\in\mathcal{A}} ||S(t)x - y||_H \to 0$, as $t \to \infty$.*

Global attractor: Nakao's Lemma

Lemma *(M. Nakao, 2006): Let* $\Phi(t)$ *be a nonnegative continuous function on* $[0, T)$, $T > 1$, *possibly* $T = +\infty$, *satisfying*

$$\sup_{t\leq s\leq t+1} \Phi(s)^{1+\gamma} \leq C(\Phi(t) - \Phi(t+1)) + K, \quad 0 \leq t < T-1, \tag{16}$$

for some $C > 0$, $K > 0$ *and* $\gamma > 0$. *Then we have*

$$\Phi(t) \leq \left[C^{-1}\gamma(t-1)^{+} + (\sup_{0\leq s\leq 1} \Phi(s))^{-\gamma}\right]^{-1/\gamma} + K^{1/1+\gamma} \tag{17}$$

for $0 \leq t < T$. *If (16) holds with* $\gamma = 0$ *then we have, instead of (17)*

$$\Phi(t) \leq \sup_{0\leq s\leq 1} \Phi(s) \left(\frac{C}{1+C}\right)^{[t]} + K, \quad 0 \leq t < T. \tag{18}$$

Here $\beta^{+} = \max\{\beta, 0\}$ *and* $[t]$ *is the biggest integer less than or equal to* t.

Global attractor: Absorbing set

Some notation: $\tilde{E}(t) = E(t) + \frac{4}{\alpha}\,||f||^2_{\ell^2}, \quad F(t)^2 = E(t) - E(t+1),$
$J(t) = \sum_{n=-\infty}^{+\infty} V(n)\dot{u}_n(t)g(\dot{u}_n(t)), \forall t \geq 0.$ We have $F(t)^2 = \int_t^{t+1} J(s)\,ds$ and

$$\tilde{E}(t) \geq \alpha_0 \|(u(t), \dot{u}(t))\|^2_H, \ \forall t \geq 0, \tag{19}$$

where $\alpha_0 = \min\left\{\frac{1}{2}, \frac{\alpha}{4}\right\}$.
Furthermore, we can choose t_1, t_2 such that

$$\int_t^{t+\frac{1}{4}} J(s)\,ds = \frac{1}{4}J(t_1) \quad \text{and} \quad \int_{t+\frac{3}{4}}^{t+1} J(s)\,ds = \frac{1}{4}J(t_2). \tag{20}$$

Remark: Inequality (19) is crucial to prove the existence of absorbing set using Nakao's lemma.

Global attractor: Absorbing set (cont.)

Lemma 2.1 *Suppose that $u_0 = (u_{0,n})$, $u_1 = (u_{1,n})$, $f = (f_n)$ belong to ℓ^2 and h, V and g satisfy hypotheses (H). Then, there exists a positive constant C_0 such that*

$$\sup_{t \le s \le t+1} \tilde{E}(s) \le C_0 \left(1 + F(t)^2\right) \left[\tilde{E}(t) - \tilde{E}(t+1)\right] + C_0 \, \|f\|_{\ell^2}^2, \; \forall t \ge 0. \tag{21}$$

Proof. Multiplying equation (1) by u_n, summing in $\mathbb{Z}$ and integrating the result in $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} \|D^p u\|_{\ell^2}^2 \, ds + \alpha \int_{t_1}^{t_2} \|u\|_{\ell^2}^2 \, ds + \int_{t_1}^{t_2} (u, h(u))_{\ell^2} \, ds = (\dot{u}(t_1), u(t_1))_{\ell^2}$$
$$- (\dot{u}(t_2), u(t_2))_{\ell^2} + \int_{t_1}^{t_2} \|\dot{u}\|_{\ell^2}^2 \, ds - \int_{t_1}^{t_2} (u, V(\cdot)\, g(\dot{u}))_{\ell^2} \, ds + \int_{t_1}^{t_2} (f, u)_{\ell^2} \, ds. \tag{22}$$

We estimate each term of the right-hand side of (22), etc...

Global attractor: Absorbing set (cont.)

Lemma 2.2 *Assume the same hypotheses of Lemma 2.1 and let* $\rho > 0$*. If* $\|(u_0, u_1)\|_H \leq \rho$*, then there exist positive constants* C_1 *and* ν *depending on* ρ *and* $\|f\|_{\ell^2}$ *such that*

$$\tilde{E}(t) \leq C_1 \, e^{-\nu t} + C_0 \, \|f\|_{\ell^2}^2, \; \forall t \geq 0.$$

where C_0 *is as in Lemma 2.1 .*

Global attractor: Absorbing set (cont.)

Proof. After estimating $F(t)^2$ in (22) we obtain

$$\sup_{t\leq s\leq t+1} \tilde{E}(s) \leq C_{1,1} \left[\tilde{E}(t) - \tilde{E}(t+1)\right] + C_0 \|f\|_{\ell^2}^2,$$

where $C_{1,1}$ is a positive constant which depends on ρ and $\|f\|_{\ell^2}$.

Applying Nakao´s Lemma with $\varphi(t) = \tilde{E}(t)$, we conclude the proof of Lemma 2.2. □

Remark: $\nu = \log(\frac{C_{1,1}+1}{C_{1,1}})$.

Global attractor: Absorbing set (cont.)

Lemma 2.3 *Under the same hypotheses of Lemma 2.1, there exists* $r_0 > 0$ *such that* $B(0; r_0) = \{(w, z) \in H;\ \|(w, z)\|_H \le r_0\}$ *is an absorbing set for* $\{S(t)\}_{t\ge 0}$ *in H.*

Global attractor: Absorbing set (cont.)

Proof. Let B any bounded set in H and let $\rho = \rho(B)$ a positive constant such that $\|(w,z)\|_H \leq \rho, \forall (w,z) \in B$. Suppose that $(u_0, u_1) \in B$. Using (19) and Lemma 2.2 we obtain

$$\|S(t)(u_0,u_1)\|_H^2 \leq \alpha_0^{-1}\tilde{E}(t) \leq \alpha_0^{-1}C_1 e^{-\nu t} + \alpha_0^{-1}C_0\|f\|_{\ell^2}^2,\ \forall t \geq 0.$$

Then, choosing $r_0 = 2\alpha_0^{-1/2}C_0^{1/2}\|f\|_{\ell^2}$ and
$\tau = \tau(B) = \max\left\{0, \frac{1}{\nu}\log\left(\frac{C_1}{C_0\|f\|_{\ell^2}^2}\right)\right\}$, we infer that $\|S(t)(u_0,u_1)\|_H \leq r_0$, $\forall t \geq \tau$.

Global attractor: Asymptotic compactness

In order to prove that $\{S(t)\}_{t\geq 0}$ is asymptotically compact it is sufficient to prove the following lemma (estimate of the tail end of the solution)[Wang, B., Physica D, vol.128, 1999, pp.41-52; S. Zhou, Journal of Math. Physics 43(1), 2002,452-465].

Lemma 2.4 *Assume the same hypotheses of Lemma 2.1. Then, for each* $\varepsilon > 0$*, there exist* $\tau(\varepsilon)$ *and* $k(\varepsilon) \in \mathbb{Z}$ *such that*

$$\sum_{|n|\geq k(\varepsilon)} \left[(\dot{u}_n(t))^2 + (u_n(t))^2\right] \leq \varepsilon, \ \forall t \geq \tau(\varepsilon).$$

Global attractor: Asymptotic compactness

To prove Lemma 2.4 we proceed as in the proof of existence of absorbing set. Let $\theta \in C^1(\mathbb{R}^+;\mathbb{R})$ such that $\theta \equiv 0$ on $[0,1]$, $\theta \equiv 1$ on $[2,+\infty)$ and $0 \le \theta \le 1$ and let $M_0 = \sup_{s\ge 0} |\theta'(s)| < +\infty$. Let $w = (w_n(t))$, where $w_n(t) = \theta\left(\frac{|n|}{k}\right) u_n(t)$, with $k \in \mathbb{Z}$ fixed. We define

$$
\begin{aligned}
E_\theta(t) =& \frac{1}{2}\sum_{n=-\infty}^{+\infty} \theta\left(\frac{|n|}{k}\right)\left[(\dot u_n)^2 + (D^p u_n)^2 + \alpha (u_n)^2\right] + \sum_{n=-\infty}^{+\infty} \theta\left(\frac{|n|}{k}\right)\tilde h(u_n) \\
&- \sum_{n=-\infty}^{+\infty} \theta\left(\frac{|n|}{k}\right) f_n u_n,
\end{aligned}
$$

$$
\tilde E_\theta(t) = E_\theta(t) + \frac{4}{\alpha}\sum_{n=-\infty}^{+\infty} \theta\left(\frac{|n|}{k}\right) f_n^2 \quad \text{and} \quad F_\theta(t)^2 = E_\theta(t) - E_\theta(t+1), \quad \forall t \ge 0.
$$

Note that $F_\theta(t)^2 = \int_t^{t+1} J_\theta(s)ds$, where $J_\theta(t) = \sum_{n=-\infty}^{+\infty} \theta\left(\frac{|n|}{k}\right) \dot u_n(t) V(n) g(\dot u_n(t))$, for all $t \ge 0$.

Differentiating $E_\theta(t)$ with respect to t, (using a lemma) we obtain

$$\frac{d}{dt}E_\theta(t) = -J_\theta(t) - \sum_{n=-\infty}^{+\infty}(\partial^+\theta_n)z_{p,n}(t), \quad \forall t \geq 0, \tag{23}$$

where $z_{p,n}(t)$ satisfies

$$\sum_{n=-\infty}^{+\infty}|z_{p,n}(t)| \leq C(p)\|(u(t),\dot{u}(t))\|_H^2, \quad \forall t \geq 0. \tag{24}$$

Moreover, we have

$$\tilde{E}_\theta(t) \geq \alpha_0 \sum_{n=-\infty}^{+\infty}\theta\left(\frac{|n|}{k}\right)\left[u_n(t)^2 + \dot{u}_n(t)^2\right], \quad \forall t \geq 0, \tag{25}$$

where as before $\alpha_0 = \min\left\{\frac{1}{2}, \frac{\alpha}{4}\right\}$.

Global attractor: Asymptotic compactness (cont.)

Lemma 2.5 *Assume the same hypotheses of Lemma 2.1 and also that* $(u_0, u_1) \in B(0; r_0)$*. Then, there exist positive constants* $C_{2,i}, i = 1, 2, 3$*, with* $C_{2,1}$ *and* $C_{2,3}$*depending on* r_0*, such that*

$$\sup_{t \leq s \leq t+1} \tilde{E}_\theta(s) \leq C_{2,1} |\tilde{E}_\theta(t) - \tilde{E}_\theta(t+1)| + C_{2,2} \|f\|_\theta^2 + \frac{C_{2,3}}{k}, \quad \forall t \geq \tau_0, \qquad (26)$$

Remark: In Lemma 2.5 $\tau_0 > 0$ is such that $S(t)B(0; r_0) \subset B(0; r_0)$, $\forall t \geq \tau_0$.
etc,.....

Global attractor: Asymptotic compactness (cont.)

Lemma 2.6 *Assume the same hypotheses of Lemma 2.1. Then, the semigroup* $\{S(t)\}_{t\geq 0}$ *is asymptotically compact in H.*

Proof. The proof is similar to that of Lemma 4.2 in S. Zhou (2002): □

The proof of Theorem 2.1. follows from Lemmas 2.3, 2.6 and Theorem 1.1, p.23, in Teman´s book [4] (Infinite dimensional dynamical systems in Mechanics and Physics, Applied Math. Sciences 68, Springer-Verlag(1988)).

Comments:

1. The existence of attractor for second order damped lattices as (1) with $p = 1$, $V(n) \equiv 1$ and a possible extra term $\gamma \Delta \dot{u}_n$ was studied in [5] (see also [6]) under more restrictive conditions. He assumed $g \in C^1(\mathbb{R};\mathbb{R}), g(0) = 0\,, 0 < c_1 \leq g'(s) \leq c_2, \forall s \in \mathbb{R}$ and a suitable relation on the parameters α, γ, c_1 and c_2.

S. Zhou, Journal of Math. Physics 43(1), 2002,452-465.

Comments (cont):

2. Ours results are still valid for more general nonlinear damping $\tilde{g}(n, \dot{u}_n)$. For example:

$$\tilde{g}(n,s) = \begin{cases} V(n)\,|s|^r s, & \text{if } |n| \leq n_0,\ s \in \mathbb{R} \\ V(n)g(s), & \text{if } |n| > n_0, s \in \mathbb{R} \end{cases}$$

with $V(n)$, g and n_0 as before and $r > 0$.

In this case we use the first part of Nakao´s Lemma.

Comments (cont):

3. The methods of this work can be used to study periodic lattices, for example, ([3]: Oliveira, J. C., Pereira, J. M., Perla Menzala, G.)

$$\ddot{u}_n - \Delta u_n = h(u_{n-1} - u_n) + h(u_{n+1} - u_n) + g(\dot{u}_{n-1} - \dot{u}_n) + g(\dot{u}_{n+1} - \dot{u}_n) + f_n \tag{27}$$

with initial conditions

$$u_n(0) = a_n \,, \quad \dot{u}_n(0) = b_n \tag{28}$$

and periodicity conditions

$$u_0(t) = u_N(t), \quad u_{n+N}(t) = u_n(t) \quad \text{for all} \quad t \geq 0 \tag{29}$$

and $(n = 1, 2, \dots)$
Remark: Equation is a semi discrete version of
$u_{tt} - u_{xx} = (h(u_x))_x + (g(u_{xt}))_x + f$.

References:

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Proof of Lemma 2.6

We will use the following notation: For $w = (w_1, w_2) \in H$, with $w_i = (w_{i,n}), i = 1, 2$ and a positive integer k, we write $\sum_{|n| \leq k} \|(w)_n\|_H^2 = \sum_{|n| \geq k} (w_{1,n}^2 + w_{2,n}^2)$. Analogously, if $|n| \geq k$.

Let $(w_m)_{m \in \mathbb{N}}$ a bounded sequence in H, such that $\|w_m\|_H \leq \mu, \ \forall m \in \mathbb{N}$ and let $t_m \to +\infty$. First, we will show that the sequence $(S(t_m)w_m)_{m \in \mathbb{N}}$ is weakly relatively compact in H. By Lemma 2.3, there exists a $\tau_\mu > 0$ such that $S(t)w_m \in B(0; r_0), \ \forall t \geq \tau_\mu$. Since $t_m \to +\infty$, then there exists an integer $M_1 = M_1(\mu) > 0$ such that $t_m > \tau_\mu, \ \forall m \geq M_1$, and

$$S(t_m)w_m \in B(0; r_0), \forall m \geq M_1. \tag{30}$$

Proof of Lemma 2.6 (cont.)

Therefore, there exists a subsequence of $(S(t_m)w_m)_{m\in\mathbb{N}}$, which we will still denote by $(S(t_m)w_m)_{m\in\mathbb{N}}$, and $w_0 \in H$ such that $S(t_m)w_m \to w_0$ weakly in H. Now, let us show that this convergence is strong in H, that is, the sequence $(S(t_m)w_m)_{m\in\mathbb{N}}$ is, in fact, relatively compact in H. Given $\varepsilon > 0$, by Lemma 2.4 , we can select a positive constant $\tau(\varepsilon)$ and a positive integer $k_1(\varepsilon)$ such that

$$\sum_{|n|\geq k_1(\varepsilon)} \|(S(t)S(\tau_\mu)w_m)_n\|_H^2 < \frac{\varepsilon^2}{8}, \ \forall t \geq \tau(\varepsilon).$$

Proof of Lemma 2.6 (cont.)

Since $t_m \to +\infty$, then $t_m \geq \tau_\mu + \tau(\varepsilon)$, $\forall m \geq M_2$, for some some positive integer $M_2 = M_2(\mu, \varepsilon)$ and hence

$$\sum_{|n| \geq k_1(\varepsilon)} \|(S(t_m)w_m)_n\|_H^2 = \sum_{|n| \geq k_1(\varepsilon)} \|(S(t_m - \tau_\mu)S(\tau_\mu)w_m)_n\|_H^2 < \frac{\varepsilon^2}{8}, \ \forall m \geq M_2. \tag{31}$$

Moreover, since $w_0 \in H$, there exists a positive integer $k_2(\varepsilon)$ such that

$$\sum_{|n| \geq k_2(\varepsilon)} \|(w_0)_n\|_H^2 < \frac{\varepsilon^2}{8}. \tag{32}$$

Proof of Lemma 2.6 (cont.)

Let $k(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. Since the sequence $(S(t_m)w_m)_{|n|\leq k(\varepsilon)}$ converges strongly to $(w_n)_{|n|\leq k(\varepsilon)}$ in $\mathbb{R}^{2k(\varepsilon)+1} \times \mathbb{R}^{2k(\varepsilon)+1}$, as $m \to +\infty$, then there exists a positive integer $M_3 = M_3(\varepsilon)$ such that

$$\sum_{|n|\leq k(\varepsilon)} \|(S(t_m)w_m - w_0)_n\|_H^2 < \frac{\varepsilon^2}{2}, \ \forall m \geq M_3. \tag{33}$$

Therefore, choosing $M = \max\limits_{1\leq i\leq 3}\{M_i\}$ and using (31)-(33) we have

$$\begin{aligned}
\|S(t_m)w_m - w_0\|_H^2 &= \sum_{|n|\leq k(\varepsilon)} \|(S(t_m)w_m) - w_0)_n\|_H^2 \\
&\quad + \sum_{|n|>k(\varepsilon)} \|(S(t_m)w_m - w_0)_n\|_H^2 \\
&\leq \frac{\varepsilon^2}{2} + 2\sum_{|n|>k(\varepsilon)} \|(S(t_m)w_m)_n\|_H^2 + 2\sum_{|n|>k(\varepsilon)} \|(w_0)_n\|_H^2 < \varepsilon^2.
\end{aligned}$$

This proves Lemma 2.6. □