Some remarks about the controllability of the Keller-Segel system

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Chemotaxis: *description of the change of motion when a population formed of individuals (such as amoebae, bacteria, endothelial cells, etc.) reacts in response (taxis) to an external chemical stimulus spread in the environment where they reside.*

- tumor angiogenesis
- inflammatory response
- wound healing
- embryology
- bacterial colony growth, pattern formation
- ...

The (one species) Keller-Segel system of chemotaxis:

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in} \quad Q := \Omega \times (0,T), \\ \epsilon v_t - \Delta v = au - bv & \text{in} \quad Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma := \partial\Omega \times (0,T), \\ u(x,0) = u_0; \; v(x,0) = v_0 & \text{in} \quad \Omega, \end{array} \right. \tag{1}$$

where $a, b > 0$ and ϵ is a small positive parameter, which is intended to go to zero.

Here:

- $u = u(x,t) \geq 0$: population density;
- $v = v(x,t) \geq 0$: density of the chemical.

We assume:

$$\Omega \subset \mathbb{R}^N, \; N = 2, 3.$$

Note that, for all $t > 0$, we have the mass preservation:

$$\int_{\Omega} u(x,t)dx = \int_{\Omega} u_0(x)dx. \qquad (2)$$

An important aspect of system (1) is the expected onset of chemotactic collapse. In other words, under suitable circumstances, the whole population concentrate in a single point (spora) in finite time.
In mathematical terms, this means formation of a Dirac delta-type singularity in finite time, i.e.

$$u(x,t) \longrightarrow M\delta(x_0) \;\text{ as }\; t \longrightarrow T, \tag{3}$$

for some $T < \infty$, where x_0 is the point where the spora develops, and

$$M = \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx.$$

Singularities (infinite mass at a point in space) occur in finite time for large data, while smooth solutions exist globally for small data.

The control problem

Given $\epsilon > 0$, $T > 0$ and $(\bar{u}, \bar{v})$, solution of (1), find a control g^ϵ such that the associated solution (u, v) to

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in } \; Q, \\ \epsilon v_t - \Delta v = au - bv + g^\epsilon 1_\omega & \text{in } \; Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \; \Sigma \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in } \; \Omega, \end{array} \right. \tag{4}$$

satisfies

$$u(T) = \bar{u}(T), \;\; v(T) = \bar{v}(T) \tag{5}$$

and

$$g^\epsilon \text{ is bounded independently of } \epsilon. \tag{6}$$

Note that (5) implies:

$$\int_\Omega u_0(x)dx = \int_\Omega \bar{u}_0(x)dx.$$

The importance of (6) is due to the fact that in many applications[1] system (1) is approximated, when $\epsilon \to 0^+$, by

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u\nabla v) & \text{in } \; Q, \\ -\Delta v = au - bv & \text{in } \; Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \; \Sigma, \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in } \; \Omega. \end{array}\right. \tag{7}$$

[1] See [1] and [7].

Our result

Theorem (Chaves-Silva & Guerreo, 2014)

Let $0 < \epsilon \leq 1$ and $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$. Then, there exists $\delta > 0$ such that, for any $(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ with $u_0, v_0 \geq 0$, satisfying $\int_\Omega u_0 dx = M_1$, $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$ and $||(u_0 - M_1, v_0 - M_2)||_{H^1(\Omega)\times H^2(\Omega)} \leq \delta$, there exists $g^\epsilon \in L^2(0, T; H^1(\Omega))$, with $\|g^\epsilon\|_{L^2(0,T;H^1(\Omega))}$ bounded independently from ϵ, such that the associated solution (u, v) to (1) satisfies:

$$(u(T), v(T)) = (M_1, M_2) \text{ in } \Omega.$$

How do we do?

We linearize system (4) around (M_1, M_2):

$$\left| \begin{array}{ll} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{in} \;\; Q, \\ \epsilon v_t - \Delta v = au - bv + g^{\epsilon}\chi & \text{in} \;\; Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \;\; \Sigma, \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in} \;\; \Omega, \end{array} \right. \tag{8}$$

where h_1 is a given exterior force belonging to an appropriate Banach Space and having exponential decay at $t = T$.

The previous Theorem is then a consequence of:

Theorem (2)

Let $0 < \epsilon \leq 1$ *and* $(M_1, M_2) \in \mathbb{R}^2$ *be such that* $aM_1 - bM_2 = 0$*. Assume that:*

$$(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega), \int_\Omega u_0 dx = 0, \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{9}$$

and

$$h_1 \text{ has "good" exponential decay at } t = T. \tag{10}$$

Then, there exists a control $g^\epsilon \in L^2(0, T; H^1(\Omega))$*, bounded independently of* ϵ*, such that, if* (u, v) *is the associated solution to* (8)*, then*

$$u(T) = v(T) = 0.$$

Theorem (2) is proved through Carleman inqualities for the adjoint system of (8).

The adjoint system of (8) reads

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\xi + f_1 & \text{in } \; Q, \\ -\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi + f_2 & \text{in } \; Q, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{on } \; \Sigma, \\ \varphi(x,T) = \varphi_T; \; \xi(x,T) = \xi_T & \text{in } \; \Omega, \\ \int_\Omega \varphi_T(x)dx = 0. & \end{array}\right. \tag{11}$$

We prove:

Theorem

Given $0 < \epsilon \leq 1$, there exists $C = C(\Omega, \omega)$ such that for any $(\varphi_T, \xi_T) \in L^2(\Omega)^2$ and any $f_1, f_2 \in L^2(Q)$, the solution (φ, ξ) of system (11) satisfies

$$\iint_Q e^{C/t^m}|\Delta\varphi|^2 dxdt + \iint_Q e^{C/t^m}|\xi|^2 dxdt \leq C\Big(\iint_{\omega\times(0,T)} e^{C/t^m}|\xi|^2 dxdt + \iint_Q e^{C/t^m}(|f_1|^2 + |f_2|^2)dxdt, \tag{12}$$

for some $m > 0$.

Comments

- We are able to avoid the blow up for regular data.
- We know nothing about the controllability of the Keller-Segel system (1) around non constant trajectories, even for a fixed ϵ. The linearization around $(\bar{u}, \bar{v})$ reads

$$\left| \begin{array}{ll} u_t - \Delta u = \nabla \cdot (\bar{u} \nabla v) + \nabla \cdot (u \nabla \bar{v}) + h_1 & \text{in} \quad Q, \\ \epsilon v_t - \Delta v = au - bv + g^{\epsilon} \chi & \text{in} \quad Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in} \quad \Omega. \end{array} \right. \tag{13}$$

Other related systems

Keller-Segel + Stokes:

$$\left| \begin{array}{ll} u_t + y \cdot \nabla u - \Delta u = \nabla \cdot (u \nabla v) & \text{in} \quad Q, \\ v_t + y \cdot \nabla v - \Delta v = au - bv & \text{in} \quad Q, \\ y_t - \Delta y + \nabla \Pi = 0 & \text{in} \quad Q \\ \nabla \cdot y = 0 & \text{in} \quad Q. \end{array} \right. \tag{14}$$

Two species Keller-Segel:

$$\left| \begin{array}{ll} u^1_t - \Delta u^1 = -\nabla \cdot (u^1 \nabla v) & \text{in} \quad Q, \\ u^2_t - \Delta u^2 = -\nabla \cdot (u^2 \nabla v) & \text{in} \quad Q, \\ v_t - \Delta v = a^1 u^1 + a^2 u^2 - bv & \text{in} \quad Q. \end{array} \right. \tag{15}$$

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