A Stable Equal Order Finite Element Discretization of the Shallow Water Equations of the Ocean

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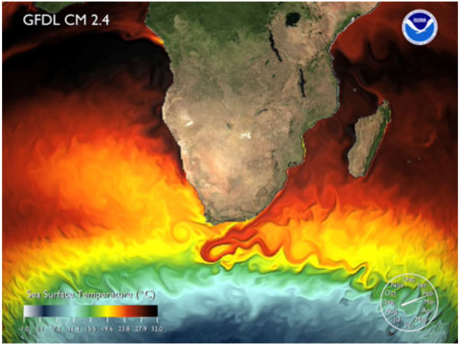
Characteristics of the Earth's Oceans



Simulated Sea Surface Temperature

- Absorbs energy from the Sun and stores it.
- Transports heat from the equator towards the poles.
- 71% of Eath's surface is covered by the oceans.
- 1000 times the heat capacity of the atmosphere.
- Most of the Ocean's KE is contained in meso-scale eddies ($<$100km).

Challenges

- Complex domain, coastlines and undersea mountain ranges.
- Small spatial scales, yet long time scales.
- Long memory, due to heat capacity and inertia, requiring several thousand simulated years for “spin up.”
- 0.1° resolution or higher needed to capture the bulk of the energy contained in the meso-scale eddy field.
- Large amounts of data, $\sim$1TB per simulated year for 0.1° grid resolution.

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Shallow Water Equations (SWE)

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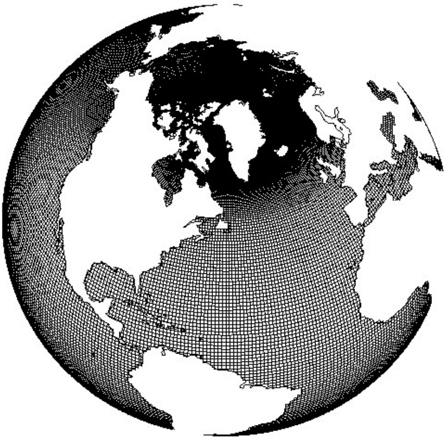
$$\begin{aligned} \eta_t + \Theta^{-1} H \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} + Ro^{-1} \mathbf{u}^{\perp} + Fr^{-2} \Theta \nabla \eta - Re^{-1} \Delta \mathbf{u} = \mathbf{0} \end{aligned} \qquad \text{on } \Omega \qquad (1)$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0} \qquad \text{on } \delta\Omega \qquad (2)$$

Why finite elements?



Finite Difference grid of GIOMAS

Finite Element mesh of SLIM

Some Known Issues with Finite Elements

- Mathematically sophisticated (Good for Mathematicians bad for Non-Mathematicians).
- Complicated to program.
  - Use packages such as FEniCS, FreeFEM, OpenFOAM, etc.
- Spurious computational modes for certain finite element pairs. (similar problem with finite differences)
  - Use a different formulation of the problem, e.g. Vorticity-Stream function form.
  - Use Taylor-Hood or lesser known elements such as $P_1 - P_1^{NC}$.
  - Use a stabilization scheme.

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cG(1)cG(1) Finite Element

- Spatial Discretization
  - Trial Functions - Piecewise linear
  - Test Functions - Piecewise linear
- Temporal Discretization
  - Trial Functions - Piecewise linear
  - Test Functions - Piecewise constant
- Weighted least squares stabilization

Discretization of SWE

$$\begin{aligned} k_n^{-1}(\mathbf{u}_n - \mathbf{u}_{n-1}, \mathbf{v}) + Ro^{-1}(\bar{\mathbf{u}}^{\perp}, \mathbf{v}) - Fr^{-2}\Theta\,(\bar{\eta}, \nabla\cdot\mathbf{v}) \\ + k_n^{-1}(\eta_n - \eta_{n-1}, \chi) + H(\nabla\cdot\bar{\mathbf{u}}, \chi) \\ + \delta_1(R_1(\bar{\mathbf{u}}_h^n, \eta_h^n), R_1(\mathbf{v}, \chi)) \\ + \delta_2(R_2(\bar{\mathbf{u}}_h^n, \eta_h^n), R_2(\mathbf{v}, \chi)) \end{aligned} \quad (3)$$

where

$$\bar{\mathbf{u}}_h^n = \frac{1}{2}(\mathbf{u}_h^n + \mathbf{u}_h^{n-1}), \quad \bar{\eta}_h^n = \frac{1}{2}(\eta_h^n + \eta_h^{n+1})$$

and

$$\begin{aligned} R_1(\mathbf{v}, \chi) &= (\bar{\mathbf{u}}\cdot\nabla)\,\mathbf{v} + Ro^{-1}\mathbf{v}^{\perp} + Fr^{-2}\Theta\nabla\chi \\ R_2(\mathbf{v}, \chi) &= \Theta^{-1}\nabla\cdot\mathbf{v} \end{aligned}$$

are the linearized strong residuals while

$$\delta_1 = \frac{Ro\,Fr^2\,\Theta^{-1}}{2}(k_n^{-2} + |\mathbf{u}^n|^2 h_n^{-2})^{-1/2}, \quad \delta_2 = \frac{\Theta}{2}(k_n^{-2} + |\eta^n|^2 h_n^{-2})^{-1/2}.$$

Linear Inviscid SWE

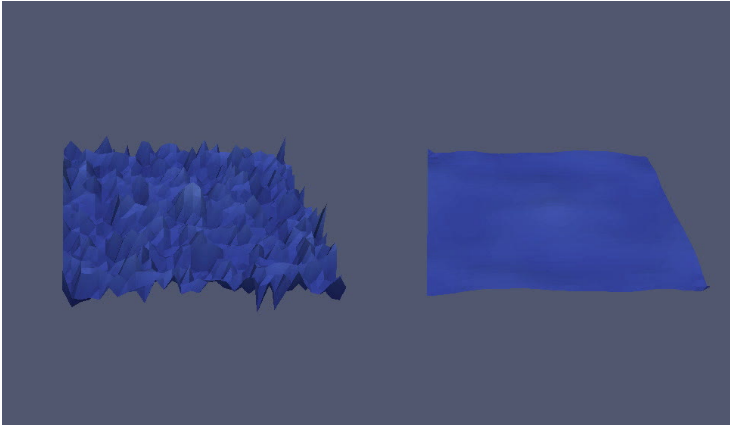
- Compare the standard $P_1 - P_1$ finite element pair to cG(1)cG(1) applied to the Linear Inviscid SWE, i.e.

$$\begin{aligned} \eta_t + \Theta^{-1} H \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}_t + Ro^{-1}\mathbf{u}^{\perp} + Fr^{-2}\Theta\nabla\eta &= \mathbf{0} \end{aligned} \quad \text{on } \Omega \tag{4}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{0} \qquad \text{on } \delta\Omega \tag{5}$$

- $Ro = 0.1$
- $Fr = 0.1$
- $\Theta = 1$
- $H = 1.63$
- Initial Condition:

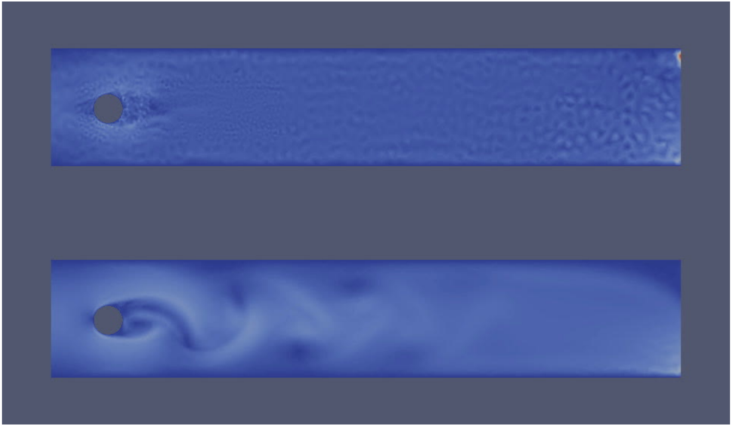
$$\begin{aligned} \mathbf{u}_0 &= \mathbf{0} \\ \eta_0 &= A\, e^{-(x_0^2 + x_1^2)/(2*\sigma^2)}, \\ A &= 1.0, \sigma = 5 \times 10^{-2} \end{aligned} \tag{6}$$



Simulated Gaussian Drop for Linear Inviscid SWE, Height
Left:$P_1 - P_1$, Right: cG(1)cG(1)

Flow Around an Island

- Compare the standard $P_1 - P_1$ finite element pair to cG(1)cG(1)
- $Re = 1\,000$
- $Ro = 0.1$
- $Fr = 0.1$
- $\Theta = 1$
- $H = 1.63$
- $\eta = 1$ at inflow and $\eta = 0$ at outflow.
- $(\mathbf{u}_0, \eta_0) = (\mathbf{0}, 0)$



Simulated flow around an Island for SWE, Velocity
Top: $P_1 - P_1$, Bottom: cG(1)cG(1)

Questions?