TWO INVERSE PROBLEMS FOR THE STOKES EQUATIONS AND THE TURNPIKE PROPERTY IN THE CONTEXT OF OPTIMAL DESIGN

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Isakov Result

Consider the parabolic problem

$$\begin{cases} \partial_t u - \text{div } (a\nabla u) + cu &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ u &= g_0 \quad , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= 0 \quad , \quad \text{in } \Omega, \end{cases}$$

We define the Dirichlet-to-Neumann map as

$$\Gamma_l(g_0) = \frac{\partial u}{\partial n}, \qquad \text{on } \partial\Omega$$

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- The proof is based in make use of stabilization of solutions of parabolic problem when $t \to \infty$, reducing the inverse parabolic problem to inverse elliptic problem with parameter.

Stokes Equations

$$(1) \qquad \begin{cases} u_t - \text{div}\,(\sigma_\mu(u,p)) = 0 & , \quad \text{in } \Omega \times (0,T), \\ \text{div}\, u = 0 & , \quad \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & , \quad \text{in } \Omega, \end{cases}$$

where $u = (u_1, u_2, u_3)$ is the velocity vector field and p is the pressure and

$$\sigma_\mu(u,p) = 2\mu e(u) - pI_3$$

is the stress tensor, where $e(u) = ((\nabla u) + (\nabla u)^T)/2$, and $\mu > 0$ is the viscosity function.

Mathematical Setup

Let $\Omega \subset \mathbb{R}^3$ be an open bounded connected domain with boundary $\partial\Omega \in C^2$.

Consider the following boundary value problem

$$(2) \qquad \begin{cases} u_t - \operatorname{div}\,(\sigma_\mu(u,p)) &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ \operatorname{div}\, u &= 0 \quad , \quad \text{in } \Omega \times (0,T), \\ u &= g \quad , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad , \quad \text{in } \Omega, \end{cases}$$

where g satisfies the compatibility condition

$$\int_{\partial\Omega} g \cdot nds = 0,$$

where n is the unit outer normal of $\partial\Omega$,

- Assume that the solution of (2) exists and the trace

$$\sigma_\mu(u,p)\cdot n|_{\partial\Omega}$$

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- Define the set of Cauchy data for (2)

$$S_\mu = \{(u, \sigma_\mu(u,p)\cdot n)|_{\partial\Omega\times(0,T)} \ : \ (u,p) \text{ solution to (2)}\}.$$

Inverse Problem : Determine μ from S_{μ}.

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Identifiablity : if $S_{\mu_1} = S_{\mu_2}$ then $\mu_1 = \mu_2$? .

- Heck, Li, and Wang (2007): Identifiability for Stationary Stokes Equation.
- Lai, Uhlmann, and Wang (2014) : Identifiability for Stokes and Navier-Stokes equations in the plane.
- Isakov (1999) : Some inverse problem for the diffusion equation.

Main result

Theorem 1

Assume that μ_1 and μ_2 are two viscosity functions satisfying $\mu_1, \mu_2 \in C^k(\overline{\Omega})$ for $k \geq 8$ and

(3) $$\mu_i > 0, \ \forall i = 1, 2.$$

(4) $$\mu_1(x) = \mu_2(x), \ \forall x \in \partial\Omega.$$

Let S_{μ_1} and S_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1} = S_{\mu_2}$, then $\mu_1 = \mu_2$.

Sketch Proof

- Stabilization of solutions of following Stokes problem

$$
(5) \qquad \begin{cases} u_t - \text{div}\,(\sigma_\mu(u,p)) + u = 0 & , \quad \text{in } \Omega \times (0,T), \\ \text{div}\, u = 0 & , \quad \text{in } \Omega \times (0,T), \\ u = g & , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & , \quad \text{en } \Omega. \end{cases}
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- Identifiability result for the stationary Stokes problem

$$\begin{cases} -\mathrm{div}\,(\sigma_\mu(u_\infty, p_\infty)) + u_\infty = 0 & , \quad \text{in } \Omega, \\ \mathrm{div}\, u_\infty = 0 & , \quad \text{in } \Omega, \\ u = g(T^*) & , \quad \text{on } \partial\Omega. \end{cases}$$

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- Conclusion of result.

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Let $\Omega \subset \mathbb{R}^d$ be an open bounded connected domain with smooth boundary. Consider the following boundary value problem

$$(6) \qquad \left\{ \begin{array}{rcll} -\text{div}\,(\sigma_\mu(u,p)) & = & 0 & , \quad \text{in } \Omega, \\ \text{div}\, u & = & 0 & , \quad \text{in } \Omega, \\ u & = & g & , \quad \text{on } \partial\Omega, \end{array} \right.$$

Define the set of Cauchy data for (6)

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)\cdot n|_{\partial\Omega}) \;:\; (u,p) \text{ solution to } (6)\}.$$

Open Problem Inverse Problem : Let u_1 and u_2 be solutions of (6) corresponding to the viscosities μ_1 and μ_2 and assume that the corresponding observations on $\partial\Omega$ are close, then do we have the viscosities are close?

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$$\text{Stability : } \|\mu_1 - \mu_2\|_X \leq C\omega\left(\|S_{\mu_1} - S_{\mu_2}\|_Y\right),$$

for some function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(s) \to 0$ as $s \to 0$, and for some suitable spaces X and Y?.

Inspired by the isotropic elasticity system, see [1], consider

$$u = \mu^{-1/2} w + \mu^{-1}\nabla f - f\nabla\mu^{-1},$$

then

$$\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \mu^{-1/2} w + \mu^{-1}\nabla f - f\nabla\mu^{-1} \\ \mathrm{div}\,(\mu^{1/2} w) + 2\delta f \end{pmatrix}$$

is a solution of (6) and (w, f) satisfies

(7) $$\Delta \begin{pmatrix} w \\ f \end{pmatrix} + A_1(x) \begin{pmatrix} \nabla f \\ \mathrm{div}\ w \end{pmatrix} + A_0(x) \begin{pmatrix} w \\ f \end{pmatrix} = 0,$$

where

$$A_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} & -\mu^{-1}\nabla\mu \\ 0 & \mu^{1/2} \end{pmatrix}.$$

References

- G. Eskin and J. Ralston. *On the inverse boundary value problem for linear isotropic elasticity*, Inverse Problems, **18** (2002), 907-921.
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Turnpike Property

We consider the dynamics

$$(8) \qquad \begin{cases} \dot{x}(t) &= f(x(t), u(t)), \\ x(0) &= x_0, \end{cases}$$

and a corresponding optimal control problem

$$\min_u J^T(u) := \int_0^T f^0(x(t), u(t))dt, \qquad x \text{ solution of } (8),$$

and the stationary analogue problem

$$\min_u J_s(u) := f^0(x, u)ds, \qquad \text{with the constraint } f^0(x, u) = 0.$$

We assume that both J^T and J_s admit minimal control (and state).

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APPLICATIONS:

- In econometry: it stipulate that the solution of an optimal control problem in large time should spend most of its time near a steady-state
- Optimal shape design or optimal materials in aeronautics: the use of steady state models for optimal design is classical in aeronautics. So, these steady state optimal designs are limits as time tends to infinity of time evolution optimal desgins?

- Finite and infinite dimensional linear systems: Porretta and Zuazua (2013)
- Finite dimensional nonlinear systems: Trélat and Zuazua (2015)
- Optimal design for the heat equation: Allaire, Munch and Periago (2010)

Shape optimization

- Finite dimensional optimal design: Consider

$$\dot{x}(t) + A(t)x(t) = b$$

and consider the simple minimization criterion

$$C_T(u) = \int_0^T (\|x(t) - x^*\|^2 + \|A(t)\|^2) dt.$$

Besides, consider the analogue steady-state problem

$$Ax = b$$

and the functional to be minimized is

$$\|x - x^*\|^2 + \|A\|^2.$$

- QUESTION: The optimal time-dependent coefficients of $A_T(t)$ approximate the those of the optimal steady state one A^*, as the time-horizon T is large enough?

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- Similar questions can be formulated in the PDE setting, for example, in optimal design (shape), parabolic equations with coefficients depending in time and space.

Thank you for your attention

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Muchas Gracias