Uniform polynomial stability of C_0-Semigroups

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Outline

I- Introduction

Stability

$$(S)\begin{cases} x'(t) = Ax(t) + Bu(t) & , t \geqslant 0 \\ x(0) = x_0 \end{cases}$$

$$\xrightarrow{(u(t)=Fx(t))} (S)\begin{cases} x'(t) = Ax(t) & , t \geqslant 0 \\ x(0) = x_0 \end{cases} \Longrightarrow x(t) = T(t)x_0$$

Stability of (S)

We distinguish several forms of stability :

1. exponential stability
2. polynomial stability
3. ...

«Aucun corps ne se met en mouvement ou revient au repos par lui-même» Ibn SINA.

Exponential Stability of (S)

Gearhart. Spectral theory for contraction semigroups on Hilbert spaces. Trans. Amer. Math. Soc. 236 : 385-394, (1978).

Prüss. On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. 284 (1984), 847-857.

Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Diff. Eq., 1 (1985), 43-56.

Theorem. (Gearhart 1978, Prüss 1984, Huang 1985)

Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on the Hilbert space H. Then $(T(t))_{t\geqslant 0}$ is exponentially stable iff $(.I - A)^{-1} \in H^\infty(\mathcal{L}(X))$.

Uniform Exponential Stability of (S_n) ! !

$$(S_n)\begin{cases} x_n'(t) = A_n x_n(t) & , t \geqslant 0 \\ x_n(0) = x_n^0 \end{cases} \Longrightarrow x_n(t) = T_n(t)x_n^0$$

(S_n) *exp. stable iff* $\forall n, \exists M_n, \alpha_n : \|T_n(t)x_n^0\| \leq M_n e^{-\alpha_n t}\|x_n^0\|$

$\Longrightarrow \|T_n(t)x_n^0\| \leq M e^{-\alpha t}\|x_n^0\|, \forall n$??

- **Infante J. A. and E. Zuazua**. Boundary observability for the space semi-discretization of the 1-D wave equation, M2AN, 33, 2 (1999), pp. 407-438.
- **K. Ramdani, T. Takahashi, and M. Tucsnak**. Uniformly exponentially stable approximations for a class of second order evolution equations-application to LQR problems. ESAIM Control Optim. Calc. Var., 13(3) :503-527, 2007.

Uniform Exponential Stability of (S_n) ! !

- **S. Ervedoza and E. Zuazua**. Uniformly exponentially stable approximations for a class of damped systems, 2008.
- **S. Ervedoza, Ch. Zheng, E. Zuazua**. On the observability of time-discrete conserva- tive linear systems, 2008.
- **Sylvain Ervedoza and Enrique Zuazua**. Uniform exponential decay for viscous damped systems, 2009.

Uniform Exponential Stability of c_0-semigroups : Zhuangyi Liu and Songmu Zheng.(1993)

Let $T_n(t)$ $(n = 1, \dots)$ be a sequence of C_0-semigroups pf operators on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators. Then $T_n(t)$ are uniformly exp. stable iff the following three conditions hold :

1. $$\sup_{n\in\mathbb{N}}\{Re\lambda;\ \lambda\in\sigma(A_n)\} = \sigma_0 < 0;$$
2. $\exists\sigma \in (\sigma_0, 0)$ such that
$$\sup_{Re\lambda\geqslant\sigma, n\in\mathbb{N}}\{(\lambda I - A_n)^{-1}\} = M_0 < \infty;$$
3. $\exists M_1 > 0$ such that
$$\|T_n(t)\|_{\mathcal{L}(H_n)} \leqslant M_1 < \infty \quad \forall t > 0, \qquad n \in \mathbb{N}$$

Uniform Exponential Stability of a thermoelastic system

Zhuangyi Liu and Songmu Zheng.(1993) : Uniform Exponential Stability and Approximation in Control of a Thermoelastic System.

$$(S)\begin{cases} u_{tt} - c^2 u_{xx} + c^2\gamma\theta_x = 0 & ,(0,\pi)\times(0,+\infty), \\ \theta_t + \gamma u_{xt} - \theta_{xx} = 0 & ,(0,\pi)\times(0,+\infty), \\ u|_{x=0,\pi} = \theta|_{x=0,\pi} = 0, & t > 0. \end{cases}$$

For certain systems $u(t) \longrightarrow 0$ *polynomially* and *not exponentially* ! !

Polynomial Stability of (S)

$$(S)\begin{cases} x'(t) = Ax(t) & , t \geqslant 0 \\ x(0) = x_0 \end{cases} \Longrightarrow x(t) = T(t)x_0$$

Definition

(S) is polynomially stable iff

$$\|T(t)A^{-\alpha}x\| \leq Ct^{-1}\|x\|, \qquad \forall t > 0.$$

where $\alpha > 0,\ x \in D(A^{\alpha})$.

II- Uniform polynomial stability

Polynomial Stability of (S)

Theorem. (Borichev and Tomilov, 2009)

Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

$$
\begin{array}{lll}
(i) & \|R(is, A)\| = O(|s|^{\alpha}), & s \to \infty. \\
(ii) & \|T(t)(-A)^{-\alpha}\| = O(t^{-1}), & t \to \infty. \\
(iii) & \|T(t)(-A)^{-1}\| = O(t^{\frac{-1}{\alpha}}), & t \to \infty.
\end{array}
$$

- **Bátkai, A., Engel, K.-J., Prüss, J., Schnaubelt, R.** : Polynomial stability of operator semigroups. Math. Nachr. 279, 1425-1440 (2006).
- **Borichev Alex, Tomilov Yu**, :Optimal polynomial decay of functions and operator semigroups (2009).

Uniform polynomial stability of (S_n)

$$(S_n)\begin{cases} x_n'(t) = A_n x_n(t) & , t \geqslant 0 \\ x_n(0) = x_n^0 \end{cases} \Longrightarrow x_n(t) = T_n(t)x_n^0$$

$$\|tT_n(t)A_n^{-\alpha}\| \leq C_n, \qquad \forall t > 0, \ \forall n.$$

$$\Longrightarrow \mathbf{C_n} = \mathbf{C}??$$

Main result

Theorem (*)

Let $T_n(t)$ $(n = 1, \ldots)$ be a uniformly bounded sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that $i\mathbb{R} \subset \rho(A_n)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent :

1. $$\sup_{s,\, n\in\mathbb{N}} |s|^{-\alpha}\|R(is, A_n)\| < \infty.$$
2. $$\sup_{t\geqslant 0,\, n\in\mathbb{N}} \|tT_n(t)A_n^{-\alpha}\| < \infty.$$
3. $$\sup_{t\geqslant 0,\, n\in\mathbb{N}} \|t^{\frac{1}{\alpha}}T_n(t)A_n^{-1}\| < \infty.$$

Uniform Polynomial Stability of (S_n)

Lemma 1

Let $T_n(t)$ $(n = 1, \ldots)$ be a sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators and let $\mathbb{C}_+ := \{z \in \mathbb{C} \ : Re\, z > 0\}$. Then $T_n(t)$ is uniformly bounded iff

$$\mathbb{C}_+ \subset \rho(A_n), \ \ and$$

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}}} \xi \int_{\mathbb{R}} \left(\|R(\xi + i\eta, A_n)x\|^2 + \|R(\xi + i\eta, A_n^*)x\|^2 \right) d\eta < \infty$$

for all $x \in H_n$.

Sketch of the Proof of Lemma 1

$\Longrightarrow$

We consider the rescaled semigroup $T_n^{-\xi}(t) := e^{-\xi t} T_n(t)$.
$R(\xi + i\eta, A_n)x = \int_0^\infty e^{-i\eta t} T_n^{-\xi}(t)xdt$.
Plancherel's Theorem implies :
$\sup\limits_{\substack{\xi>0 \\ n\in\mathbb{N}}} \xi \int_{\mathbb{R}} \|R(\xi + i\eta, A_n)x\|^2 d\eta \leqslant \pi M^2 \|x\|^2$.
By symmetry we obtain the same estimate for the resolvent of A_n^*.

$\Longleftarrow$

We use the inversion formula :
$\langle T_n(t)x, x^* \rangle = \frac{1}{2\pi i t} \lim\limits_{\omega\to\infty} \int_{\tau - i\omega}^{\tau + i\omega} e^{\lambda t} \langle R^2(\tau + i\beta, A_n)x, x^* \rangle d\lambda$,
we choose $\tau = \frac{1}{t}$:
We deduce $\|T_n(t)\| \leqslant \frac{eC}{4\pi} \quad \forall t > 0,\ n \in \mathbb{N}$, with $C > 0$ a constant independent of n.

Lemma 2

Let $T_n(t)$ $(n = 1, \ldots)$ be a sequence of uniformly bounded C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that $i\mathbb{R} \subset \rho(A_n)$. Then for a fixed $\alpha > 0$, the following assertions are equivalent :

1. $$\sup_{Re\ \lambda>0,\ n\in\mathbb{N}} \frac{\|R(\lambda, A_n)\|}{1+|\lambda|^{\alpha}} < \infty.$$
2. $$\sup_{Re\ \lambda>0,\ n\in\mathbb{N}} \|R(\lambda, A_n)A_n^{-\alpha}\| < \infty.$$
3. $$\sup_{s,\ n\in\mathbb{N}} |s|^{-\alpha}\|R(is, A_n)\| < \infty.$$

Lemma 3 : Moment inequality

Let $\alpha < \beta < \gamma$, then there exists a constant $k(\alpha, \beta, \gamma) > 0$, such that the following inequality hold :
$\|A_n^{\beta}x\| \leq \mathbf{k}(\alpha, \beta, \gamma)\|A_n^{\gamma}x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}.\|A_n^{\alpha}x\|^{\frac{\gamma-\beta}{\gamma-\alpha}}, \quad \forall x \in D(A_n^{\gamma}), \ \forall n \in \mathbb{N}.$

Lemma 3 has been established in the particular case $n = 1$.

K. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, p : 141, Th : 5.34.

Similarly (1) $\Longleftrightarrow$ (2) was established for $n = 1$ by :

Huang, S.-Z., van Neerven, J.M.A.M. : B-convexity, the analytic Radon-Nikodym property and individual stability of C_0-semigroups. J. Math. Anal. Appl. 231, 1-20 (1999)

Latushkin, Yu., Shvydkoy, R. : Hyperbolicity of semigroups and Fourier multipliers, In : Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), Oper. Theory Adv. Appl., vol. 129, pp. 341-363 Birkhäuser, Basel (2001)

Sketch of the Proof of Lemma 3

First case : $\gamma = 0$, $\alpha = -\alpha_1$, $\beta = -\beta_1$ $(0 < \beta_1 < \alpha_1)$ and $p \leq \alpha_1 < p + 1$.

$$A_n^{-\beta_1} = \frac{1}{2\pi i} \int_{\Gamma_a} \lambda^{-\beta_1} R(\lambda, A_n) d\lambda,$$

where we assume that $\|R(\lambda, A_n)\| \leq \frac{M}{1+|\lambda|}$.
We can show for $\lambda = se^{\pm\pi i}$
$\|A_n^{\alpha} R^{p+1}(-s, A_n)\| \leq \frac{c}{(1+s)^{n+1-\alpha}} \qquad (p \leq \alpha < p + 1)$
We show that $\|A_n^{-\beta_1} x\| \leq k(\alpha_1, \beta_1) \|x\|^{\frac{\alpha_1 - \beta_1}{\alpha_1}} . \|A_n^{-\alpha_1} x\|^{\frac{\beta_1}{\alpha_1}}$,
with $k(\alpha_1, \beta_1) = c^{\frac{\beta_1}{\alpha_1}} M^{(p+1)\frac{(\alpha_1-\beta_1)}{\alpha_1}} \left(\frac{1}{\alpha_1 - \beta_1} + \frac{1}{\beta_1} \right)$.
General case : $\alpha < \beta < \gamma$ and $x \in D(A_n^{\gamma})$.
We apply the last inequality to the element $A_n^{\gamma} x$ with $\alpha_1 = \gamma - \alpha$ and $\beta_1 = \gamma - \beta$.

Sketch of the Proof of Lemma 2

(1) $\Leftrightarrow$ (2)
$R(\lambda, A_n)$ is bounded on $D = \{\lambda \in \mathbb{C} / |\lambda| \leqslant \varepsilon\}$; S any subset of $\rho(A_n)$ s.t $D \cap S = \emptyset$.
We remark that : $\frac{R(\lambda,A_n)}{|\lambda|^\alpha} = \frac{1+|\lambda|^\alpha}{|\lambda|^\alpha}\frac{R(\lambda,A_n)}{1+|\lambda|^\alpha}$, we can show by induction :

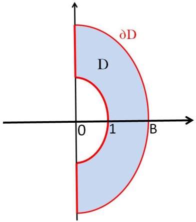
$$R(\lambda, A_n) A_n^{-p} = \frac{R(\lambda, A_n)}{\lambda^p} + \sum_{k=0}^{p-1} (-1)^p \frac{A_n^{-(p-k)}}{\lambda^{k+1}}.$$

For $\alpha = p \in \mathbb{N}$: $\|R(\lambda, A_n) A^{-\alpha}\| \leq d_\alpha \frac{\|R(\lambda,A_n)\|}{1+|\lambda|^\alpha} + c_\alpha$,
$\frac{\|R(\lambda,A_n)\|}{1+|\lambda|^\alpha} \leq \|R(\lambda, A_n) A^{-\alpha}\|$,
with $c_\alpha,\ d_\alpha > 0$ independent of n.
For $\alpha = p \notin \mathbb{N}$: we apply the moment inequality.

Sketch of the Proof of Lemma 2

(1) $\Leftrightarrow$ (3)

Apply the maximum principle to the function $F(\lambda) = R(\lambda, A_n)\lambda^{-\alpha}(1 + \frac{\lambda^2}{B^2})$ on the domain $D := \{\lambda \in \mathbb{C} : \ Re(\lambda) \geqslant 0, 1 \leqslant |\lambda| \leqslant B\}$ for large B, and to use the estimate $\|R(\lambda, A_n)\| \leq \frac{M}{Re(\lambda)}$.



III- Application

System of partially damped wave equations :

Zhuangyi Liu and Bopeng Rao.(2006) : Frequency domain approach for the polynomial stability of a system of partially damped wave equations.

$$(S)\begin{cases} u_{tt} - a\Delta u + \alpha y = 0 \ in \ \Omega, \\ y_{tt} - a\Delta y + \alpha u = 0 \ in \ \Omega, \\ u = 0, \quad on \ \Gamma_0, \qquad a\partial_\nu u + \gamma u + u_t = 0 \quad on \ \Gamma_1 \\ y = 0 \quad on \ \Gamma, \end{cases}$$

with Ω a bounded domain in $\mathbb{R}$.
$\Gamma = \Gamma_0 \cup \Gamma_1, \ \Gamma_0 \cap \Gamma_1 = \emptyset.$

Wave equation with a localized linear dissipation :

Kim Dang Phung.(2007) : Polynomial decay rate for the dissipative wave equation.

$$(S)\begin{cases} u_{tt}-\Delta u+\alpha(x)\partial_t u=0 & \textit{in } \Omega\times\mathbb{R}^+, \\ u=0 & \textit{on } \partial\Omega\times\mathbb{R}^+, \\ (u(.,\ 0),\ \partial_t u(.,\ 0))=(u_0,\ u_1) & \textit{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}$ with a boundary $\partial\Omega$ at least Lipschitz. Here, α is a nonnegative function in $L^\infty(\Omega)$ and depends on a non-empty proper subset ω of Ω on which $\frac{1}{\alpha}\in L^\infty(\omega)$ (in particular, $\{x\in\Omega;\alpha(x)>0\}$ is a non-empty open set).

Hyperbolic-parabolic coupled system :

J. Rauch, X. Zhang. Zuazua E.((2005)) : Polynomial decay for a hyperbolic-parabolic coupled system.

Zhang X., Zuazua E., Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system, J. Differential Equations 204 (2004), 2, 380-438.

$$
(S)\begin{cases}
y_t - y_{xx} = 0 & in\ (0,\infty)\times(0,1),\\
z_{tt} - z_{xx} = 0 & in\ (0,\infty)\times(-1,0),\\
y(t,1) = 0 = z(t,-1), & t\in(0,\infty),\\
y(t,0) = z(t,0), \quad y_x(t,0) = z_x(t,0) & t\in(0,\infty),\\
y(0) = y_0 & in (0,1),\\
z(0) = z_0, \quad z_t(0) = z_1 & in (-1,0).
\end{cases}
$$

Thanks For Your Attention