Uniform boundary controllability of the Schrödinger equation with vanishing viscosity

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Joint work with Ionel Rovența

One dimensional Schrödinger equation

Given any time $T > 0$ and an initial datum $y^0 \in H^{-1}(0, \pi)$, the null-controllability property of the linear 1-d Schrödinger equation in the bounded interval $(0, \pi)$,

$$\begin{cases} y_t(t,x) - i\, y_{xx}(t,x) = 0, & x \in (0,\pi),\ t > 0 \\ y(t,0) = 0,\ y(t,\pi) = v(t), & t > 0 \\ y(0,x) = y^0(x), & x \in (0,\pi) \end{cases} \tag{1}$$

consists of finding a scalar function $v \in L^2(0,T)$, called control, such that the corresponding solution y of (1) verifies

$$y(T, \cdot\,) = 0. \tag{2}$$

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  - Microlocal Analysis

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Controllability of the one dimensional heat equation

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- Moment problem
- Construction and evaluation of biorthogonals

Moment problem for the Schrödinger equation

The null-controllability of the Schrödinger equation is equivalent to solve the following moment problem:

For any $y^0 = \sum_{n=1}^{\infty} a_n \sin(nx)$, find $v \in L^2(0,T)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(s + \frac{T}{2}\right) e^{s\overline{\nu}_n} ds = \frac{(-1)^n \pi}{2ni} e^{-\frac{T}{2}\overline{\nu}_n} a_n \quad \forall\, n \geq 1, \tag{3}$$

where $\nu_n = -i\, n^2$ are the eigenvalues of the unbounded skew-adjoint differential operator corresponding to (1).

Moment problem for the Schrödinger equation

For any $T > \frac{2\pi}{\gamma_\infty}$ the following inequality holds

Ingham's inequality

$$\sum_{n\geq 1} |\alpha_n|^2 \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\geq 1} \alpha_n e^{\nu_n t} \right|^2 dt \qquad \forall\, (\alpha_n)_{n\geq 1} \in \ell^2. \quad (4)$$

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In our case (4) holds for any $T > 0$ due to the fact that

$$\gamma_\infty := \liminf_{n\to\infty} |\nu_{n+1} - \nu_n| = \infty.$$

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$$\gamma_\infty := \liminf_{n\to\infty} |\nu_{n+1} - \nu_n| = \infty.$$

Consequently, the moment problem has a solution $v \in L^2(0,T)$ for each $y^0 \in H^{-1}(0,\pi)$. This solution is not unique since the family of exponential functions $(e^{\nu_n t})_{n\geq 1}$ is not complete in $L^2(0,T)$.

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A family of functions $(\phi_m)_{m\geq 1} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ *with the property*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{\overline{\nu}_n t} dt = \delta_{mn} \quad \forall\, m, n \geq 1, \tag{5}$$

is called a biorthogonal sequence to $(e^{\nu_n t})_{n\geq 1}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

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Once we have a biorthogonal sequence to $(e^{\nu_n t})_{n\geq 1}$, it follows that a "formal" solution of the moment problem is given by

$$v(t) = \sum_{n\geq 1} \frac{(-1)^n \pi\, a_n}{2n\, i}\, \phi_n\left(t - \frac{T}{2}\right) e^{-\frac{T}{2}\,\overline{\nu}_n} \quad \forall t \in (0, T). \tag{6}$$

Moment problem for the Schrödinger equation

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$$\widehat{\phi}_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{-i\,z\,t} dt \quad \text{(Fourier transform of } \phi_m\text{)}.$$

$$\phi_m \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right) \Rightarrow \begin{cases} \widehat{\phi}_m \text{ is entire function of exponential type } \frac{T}{2} \\ \widehat{\phi}_m \in L^2(\mathbb{R}). \end{cases}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{\overline{\nu}_n\,t} dt = \delta_{mn} \Rightarrow \widehat{\phi}_m(i\overline{\nu}_n) = \delta_{mn}, \quad n \geq 1.$$

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$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\phi_m(t)e^{\overline{\nu}_n\,t}dt=\delta_{mn}\Rightarrow\widehat{\phi}_m(i\overline{\nu}_n)=\delta_{mn},\quad n\geq 1.$$

$\widehat{\phi}_m(z)$ with the above properties + Inverse Fourier Transform + Paley-Wiener Theorem $\Rightarrow \phi_m$.

Controllability of the perturbed Schrödinger equation

Our aim is to study the possibility of obtaining a control for (1) as limit of controls of the following perturbed equations

$$\begin{cases} y_t(t,x) - i\, y_{xx}(t,x) = 0, & x \in (0,\pi),\ t > 0 \\ y(t,0) = 0,\ y(t,\pi) = v(t), & t > 0 \\ y(0,x) = y^0(x), & x \in (0,\pi), \end{cases} \tag{7}$$

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$$\left\{\begin{array}{lll} y_t(t,x) - i\, y_{xx}(t,x) - \varepsilon y_{xx}(t,x) & = 0, & x \in (0,\pi),\ t > 0 \\ y(t,0) = 0,\ y(t,\pi) = v_\varepsilon(t), & & t > 0 \\ y(0,x) = y^0(x), & & x \in (0,\pi), \end{array}\right. \tag{7}$$

where ε is a small parameter devoted to tend to zero.

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$$\begin{cases} y_t(t,x) - i\, y_{xx}(t,x) - \varepsilon y_{xx}(t,x) = 0, & x \in (0,\pi),\ t>0 \\ y(t,0)=0,\ y(t,\pi) = v_\varepsilon(t), & t>0 \\ y(0,x) = y^0(x), & x \in (0,\pi), \end{cases} \tag{7}$$

where ε is a small parameter devoted to tend to zero.

Definition

A function $v_\varepsilon \in L^2(0,T)$ is called control for the initial datum y^0 in time T if the corresponding solution of (7) verifies $y(T,\cdot)=0$. If, given any $y^0 \in H^{-1}(0,\pi)$, there exists a control $v_\varepsilon \in L^2(0,T)$ for y^0 in time T, we say that (7) is null-controllable in time T.

Return

Controllability of the perturbed Schrödinger equation

We are interested in the following issue:

Given $T > 0$, $\varepsilon > 0$ and $y^0 \in H^{-1}(0, \pi)$, is there a control $v_\varepsilon \in L^2(0, T)$ for (7) such that the family $(v_\varepsilon)_{\varepsilon>0}$ converges to a control of (1) when $\varepsilon \to 0$?

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In (7), $-\varepsilon y_{xx}$ represents a viscous term. Indeed, if $v_\varepsilon = 0$,

$$\frac{d}{dt}\int_0^\pi |y(t,x)|^2 dx = -2\,\varepsilon \int_0^\pi |y_x(t,x)|^2 dx \leq 0 \quad \forall t \geq 0, \tag{8}$$

and the L^2-norm of the solution y of (7) is decreasing in time.

Controllability of the perturbed Schrödinger equation

- (7) may be viewed as a Schrödinger equation with an added viscous term. As such it models an array of optical fibers in a weakly lossy medium.

  M. Salerno, B. A. Malomed, V. V. Konotop, Shock wave dynamics in a discrete nonlinear Schrödinger equation with internal losses, Physical Review, 62 (2000), 8651-8656.

- (7) is known as the linear complex Ginzburg-Landau equation. With a cubic nonlinearity, it plays an important role in the theory of amplitude equations and provides a simple model for turbulence.

  C. D. Levermore and M. Oliver, The complex Ginzburg-Landau equation as a model problem, in: Dynamical Systems and Probabilistic Methods in PDE, Berkeley, CA, 1994, in: Lectures in Appl. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1996, 141-190.

Controllability of the perturbed Schrödinger equation

The interest of this problem is justified by the use of the vanishing viscosity as a typical mechanism to

- Study Cauchy problems
  R. J. DiPerna, Convergence of approximate solutions to conservation laws, Arch. Ration. Mech. Anal., 82 (1983), 27-70.
- Improve convergence of numerical schemes
  L. Ignat and E. Zuazua, Numerical dispersive schemes for the nonlinear Schrödinger equation, SIAM J. Numer. Anal., 47 (2009), 1366-1390.

In both examples the viscosity is devoted to tend to zero in order to obtain the original system. Thus, a legitimate question is related to the behavior and the sensitivity of the controls during this process.

Controllability of the perturbed Schrödinger equation

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3. O. Glass, A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit, Journal of Functional Analysis, 258 (2010), 852-868.
4. M. Léautaud, Uniform controllability of scalar conservation laws in the vanishing viscosity limit, preprint, 2010.

Controllability of the perturbed Schrödinger equation

- In [1] the authors pass to the limit in the dissipative wave equation to obtain a control for the heat equation.
- In [2,3] the controllability of the transport equation is consider by introducing a vanishing viscosity term. In [2] Carleman estimates are used to obtain a uniform bound for the family of controls. The same result is shown in [3], improving the control time, by means of nonharmonic Fourier analysis and biorthogonal technique.
- The recent article [4] deals with a nonlinear scalar conservation law perturbed by a small viscosity term and proves the uniform boundedness of the boundary controls.

Main result

Jump to the control problem

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Theorem

There exists $T > 0$ with the property that, for any $y^0 \in H^{-1}(0, \pi)$ and $\varepsilon \in (0, 1]$, there exists a control $v_\varepsilon \in L^2(0, T)$ of (7) such that the family $(v_\varepsilon)_{\varepsilon \in (0,1]}$ is uniformly bounded in $L^2(0, T)$ and any weak limit v of it is a control in time T for the Schrödinger equation (1).

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- $y^0 \in H^{-1}(0,\pi)$ is the optimal space.

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- Uniform controllability in time T independent of ε.
- $y^0 \in H^{-1}(0,\pi)$ is the optimal space.
- The same result should be true for arbitrary $T > 0$ (optimal time).

Proof of the main theorem: Step 1 - Moment problem

Given $y^0 = \sum_{n=1}^{\infty} a_n \sin(nx)$, its null-controllability is equivalent to solve the moment problem:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v_\varepsilon\left(s + \frac{T}{2}\right) e^{s\overline{\mu}_n} ds = \frac{(-1)^n \pi}{2n(i+\varepsilon)} e^{-\frac{T}{2}\overline{\mu}_n} a_n \quad \forall\, n \geq 1, \qquad (9)$$

where $\mu_n = \varepsilon n^2 - i\, n^2$ are the eigenvalues of the associated differential operator.

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where $\mu_n = \varepsilon n^2 - i\, n^2$ are the eigenvalues of the associated differential operator.

- The exponents μ_n are no longer purely imaginary. We cannot use Ingham's inequality.
- Can we apply the technique of Fattorini H. O. and Russell D. L.?

Proof of the main theorem: Step 2 - Biorthogonals

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Definition

A family of functions $(\zeta_m)_{m\geq 1} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta_m(t) e^{\overline{\mu}_n t} dt = \delta_{mn} \quad \forall\, m, n \geq 1, \tag{10}$$

is called a biorthogonal sequence to $(e^{\mu_n t})_{n\geq 1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

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- The element ζ_m of the biorthogonal sequence controls the one mode initial datum $y_m^0(x) = \sin(mx)$.
- If we have a finite combination of modes that we want to control, we consider a finite combination of biorthogonals.

Proof of the main theorem: Step 2 - Biorthogonals

If we have an initial datum with an infinity of modes, a "formal" solution of the moment problem is given by

$$v_\varepsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \pi a_n}{2n(i+\varepsilon)} e^{-\frac{T}{2}\overline{\mu}_n} \zeta_n\left(t - \frac{T}{2}\right) \quad \forall t \in (0,T). \qquad (11)$$

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The absolute convergence of this series depends on

- The Fourier coefficients a_n of the controlled initial datum y^0.
- The norm of the biorthogonals elements ζ_n.
- The exponentials $e^{-\frac{T}{2}\overline{\mu}_n}$ "help" the convergence in the range $n \geq \frac{1}{\sqrt{\varepsilon}}$,

$$\left| e^{-\frac{T}{2}\overline{\mu}_n} \right| = e^{-\frac{T}{2}\varepsilon n^2}.$$

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To prove the existence of a control v_ε for any $y^0 \in H^{-1}(0,\pi)$ and to get uniform bounds for its L^2-norm we need:

- Existence of biorthogonals
- Evaluation of their norms
  - uniformly bounded in the range $n < \frac{1}{\sqrt{\varepsilon}}$.
  - not too big in the range $n \geq \frac{1}{\sqrt{\varepsilon}}$.

Proof of the main theorem: Step 3 - Existence

The existence of a biorthogonal family to $(e^{\mu_n t})_{n\geq 1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ follows from the lack of completeness of the family $(e^{\mu_n t})_{n\geq 1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

Theorem (Müntz-Szász, 1916)

Let $(\lambda_n)_{n\geq 1}$ *be a sequence of complex numbers such that* $\liminf_{n\to\infty} \Re(\lambda_n) > 0$ *and* $T > 0$. *Then the set of functions* $(e^{-\lambda_n t})_{n\geq 1}$ *is complete in* $L^2(0, T)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\Re(\lambda_n)}{1 + |\lambda_n|^2} = \infty. \qquad (12)$$

In our case: $\sum_{n=1}^{\infty} \frac{\Re(\mu_n)}{1+|\mu_n|^2} = \sum_{n=1}^{\infty} \frac{\varepsilon n^2}{1+n^4+\varepsilon^2 n^4} < \infty.$

Proof of the main theorem: Step 4 - Evaluation

The evaluation of the norm of a biorthogonal family to $(e^{\mu_n t})_{n\geq 1}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is obtained from an explicit construction.

Theorem

There exists $T>0$ such that, for any $\varepsilon\in(0,1]$, we find a biorthogonal sequence $(\zeta_m)_{m\in\mathbb{N}^*}$ to the family $\left(e^{\mu_m t}\right)_{m\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ with the following property

$$||\zeta_m||_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C\,\exp\left(\alpha|\Re(\mu_m)|\right) \qquad \forall\, m\geq 1, \tag{13}$$

where C and α are positive constants independent of ε and m.

Proof of the main theorem: Step 4 - Evaluation

Proof: For any $m \in \mathbb{N}^*$, we define the function

$$P_m(z) = \prod_{\substack{n \in \mathbb{Z}^* \\ |n| \neq m}} \left(1 - \frac{z}{i\,\lambda_n}\right)\left(\frac{\lambda_n}{\lambda_n - \lambda_m}\right), \tag{14}$$

where

$$\lambda_n = \begin{cases} in + \varepsilon n, & \text{if } n = q^2 \quad \text{with } q \in \mathbb{N}^* \\ i\sqrt{1+\varepsilon^2}\, n, & \text{if } n \neq q^2,\ n > 0 \quad \text{with } q \in \mathbb{N}^* \\ \overline{\lambda}_{-n}, & \text{if } n < 0. \end{cases}$$

Note that $\lambda_{p^2} = \overline{\mu}_p$, for any $p \geq 1$. Hence, the family $(\lambda_m)_{m \geq 1}$ is "larger" than $(\mu_p)_{p \geq 1}$. This extension of the family of exponents will be very important for the behavior on the real axis of P_m.

Proof of the main theorem: Step 4 - Evaluation

Lemma

For each $m \geq 1$ *of the form* $m = p^2$ *with* $p \in \mathbb{N}^*$, P_m *is an entire function of exponential type independent of* ε *such that*

$$P_m(i\lambda_n) = \delta_{mn}, \quad n \in \mathbb{N}^*. \tag{15}$$

Proof of the main theorem: Step 4 - Evaluation

Lemma

For each $m \geq 1$ of the form $m = p^2$ with $p \in \mathbb{N}^*$, P_m is an entire function of exponential type independent of ε such that

$$P_m(i\lambda_n) = \delta_{mn}, \quad n \in \mathbb{N}^*. \tag{15}$$

Lemma

For each $m \geq 1$ of the form $m = p^2$ with $p \in \mathbb{N}^*$, the function P_m defined by (14) has the following property

$$|P_m(x)| \leq C \exp(32\varepsilon\sqrt{|x|} + 16\varepsilon\Re(\lambda_m)) \quad \forall x \in \mathbb{R}, \tag{16}$$

where C is a positive constant, independent of ε and m.

Proof of the main theorem: Step 4 - Evaluation

Lemma

For any $\varepsilon \in (0,1]$ *and* $m \geq 1$ *of the form* $m = p^2$ *with* $p \in \mathbb{N}^*$*, there exists a function* $M_{m,\varepsilon} : \mathbb{C} \to \mathbb{C}$ *such that*

1. $M_{m,\varepsilon}$ *is an entire function of exponential type independent of* ε *and* m.
2. $|M_{m,\varepsilon}(x)| \leq \exp\left(-\varepsilon\sqrt{|x|}\right)$ *for all* $x \in \mathbb{R}$.
3. $|M_{m,\varepsilon}(i\lambda_m)| \geq C\,\exp\left(-R\,|\Re(\lambda_m)|\right)$,

where C *and* R *are positive constants independent of* ε *and* m.

Proof of the main theorem: Step 4 - Evaluation

For each $m \in \mathbb{N}^*$ we define the function

$$\Psi_m(z) = P_{m^2}(z) \left(\frac{M_{m^2,\varepsilon}(z)}{M_{m^2,\varepsilon}(i\lambda_{m^2})} \right)^{32} \frac{\sin(\gamma(z - i\lambda_{m^2}))}{\gamma(z - i\lambda_{m^2})} \quad \forall z \in \mathbb{C},$$

where $\gamma > 0$ is an arbitrary constant independent of ε and m.

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where $\gamma > 0$ is an arbitrary constant independent of ε and m.

- Ψ_m is an entire function of exponential type $\frac{T}{2} > 0$.
- $\Psi_m \in L^2(\mathbb{R})$, $\|\Psi_m\|_{L^2(\mathbb{R})} \leq C \, \exp\left(\alpha|\Re(\mu_m)|\right)$.
- $\Psi_m(i\overline{\mu}_n) = \delta_{mn}$.

Proof of the main theorem: Step 4 - Evaluation

From Paley-Wiener's Theorem we deduce that there exists $(\zeta_m)_{m\geq 1} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ such that

$$\Psi_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta_m(t) e^{-izt} dx \qquad \forall m \geq 1.$$

- $(\zeta_m)_{m\geq 1} \subset L^2(-\frac{T}{2}, \frac{T}{2})$ is a biorthogonal sequence.
- $\|\zeta_m\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} = 2\pi \|\Psi_m\|_{L^2(\mathbb{R})} \leq C \exp\left(\alpha |\Re(\mu_m)|\right).$

End of the proof.

Comments

The simplest function P_m does not work. Indeed, if we choose

$$P_m(z) = \prod_{\substack{n\in\mathbb{Z}^* \\ |n|\neq m}} \left(1 - \frac{z}{i\,\overline{\mu}_n}\right)\left(\frac{\overline{\mu}_n}{\overline{\mu}_n - \overline{\mu}_m}\right), \tag{17}$$

we have an entire function which verifies the necessary relations $P_m(i\overline{\mu}_n) = \delta_{mn}$. Moreover, a product like (17), has arbitrarily small exponential type and eventually would allow to deduce a controllability result for any $T > 0$.

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we have an entire function which verifies the necessary relations $P_m(i\overline{\mu}_n) = \delta_{mn}$. Moreover, a product like (17), has arbitrarily small exponential type and eventually would allow to deduce a controllability result for any $T > 0$.
However, P_m is useless. Indeed,

$$|P_m(x)| \geq C_1 \exp\left(C_2\sqrt{|x|}\right) \quad \forall x \in \mathbb{R}$$

with C_1 and C_2 two positive constants independent of ε.

Comments

For each $\varepsilon \in (0,1]$ and $m \geq 1$, we need a multiplier $M_{m,\varepsilon}$,

$$|M_{m,\varepsilon}(x)| \leq \exp\left(-C_2\sqrt{|x|}\right) \quad \forall x \in \mathbb{R}, \tag{18}$$

$$|M_{m,\varepsilon}(i\overline{\mu}_m)| \geq \exp\left(-C_3|\Re(\mu_m)|\right) = \exp\left(-C_3\varepsilon m^2\right), \tag{19}$$

with C_3 a positive constant independent of ε.

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with C_3 a positive constant independent of ε.

$$G_{m,\varepsilon}(z) = M_{m,\varepsilon}(z)e^{(B+\eta)z}e^{C_2\sqrt{-z}}$$

From the Phragmén-Lindelöf Theorem, there exists a constant $C > 0$, independent of ε and m, such that

$$|G_{m,\varepsilon}(x+iy)| \leq C, \qquad x \leq 0,\ y \geq 0.$$

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From the Phragmén-Lindelöf Theorem, there exists a constant $C > 0$, independent of ε and m, such that

$$|G_{m,\varepsilon}(x + iy)| \leq C, \qquad x \leq 0,\ y \geq 0.$$

This gives the contradiction:

$$|M_{m,\varepsilon}(i\overline{\mu}_m)| \leq C\exp\left(-\frac{m}{\sqrt{2}}\right), \qquad m \geq 1. \tag{20}$$

Comments

- For any finite sequence $(a_n)_{n\geq 1}$, the following weighted Ingham type inequality holds

$$\sum_{n\geq 1} |a_n|^2 e^{-2\alpha|\Re(\mu_n)|} \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\geq 1} a_n e^{\mu_n t} \right|^2 dt. \qquad (21)$$

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- Other types of control problem can be addressed: punctual controls $\delta_{x_0} v(t)$ or lumped controls $v(t)g(x)$.

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- Other types of control problem can be addressed: punctual controls $\delta_{x_0} v(t)$ or lumped controls $v(t)g(x)$.
- Can we extend the result to several dimensions? Probably yes, by using Carleman estimates (work in progress).

Related problems

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} - \varepsilon \frac{u'_{j+1}(t)+u'_{j-1}(t)-2u'_j(t)}{h^2} = 0 \\ u_0(t) = 0, \;\; u_{N+1}(t) = v_h(t), \\ u_j(0) = u_j^0(x), \;\; u_j' = u_j^1(x). \end{cases} \tag{22}$$

Related problems

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u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} - \varepsilon \frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2} = 0 \\
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\end{cases}
\tag{22}
$$

- If $\varepsilon = h$ there exits a sequence of boundary controls for (22), $(v_h)_{h>0}$ which is uniformly bounded in $L^2(0,T)$ and any weak limit of it is a boundary control of the continuous wave equation.

  SM: Uniform boundary controllability of a semidiscrete 1-D wave equation with vanishing viscosity, SIAM J. Cont. Optim., 47 (2008) 2857-2885.

- If $\varepsilon = h^2$ this is no longer true.
- What is happening when $\varepsilon = h^\alpha$ with $\alpha \in (1,2)$?

Related problems

$$\begin{cases} u_t + i(-\Delta)^{\frac{1}{2}} u - \varepsilon \Delta^\alpha u = g(x) v_\varepsilon(t), \quad x \in (0,\pi),\ t > 0 \\ u(t,0) = u(t,\pi) = 0, \quad t > 0 \\ u(0,x) = u_0(x), \quad x \in (0,\pi), \end{cases} \tag{23}$$

Related problems

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- $\alpha = 1/2$: We do not have spectral controllability: Müntz-Szász condition is not satisfied.
- $\alpha = 1$: The dissipation is very strong and we have uniform controllability.

  SM, A. Pazoto, J. Ortega: Null-controllability of a hyperbolic equation as singular limit of parabolic ones, Journal of Fourier Analysis and Applications, 2011, to apper.
- $\alpha \in (1/2, 1)$? Müntz-Szász condition is still satisfied. Do we still have uniform controllability?

THANK YOU!