Uniform boundary controllability of the Schrödinger equation with vanishing viscosity

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Joint work with Ionel Rovența
Given any time \( T > 0 \) and an initial datum \( y^0 \in H^{-1}(0, \pi) \), the null-controllability property of the linear 1-d Schrödinger equation in the bounded interval \((0, \pi)\),

\[
\begin{align*}
&y_t(t, x) - i y_{xx}(t, x) = 0, \quad x \in (0, \pi), \quad t > 0 \\
y(t, 0) = 0, \quad y(t, \pi) = v(t), \quad t > 0 \\
y(0, x) = y^0(x), \quad x \in (0, \pi)
\end{align*}
\]

(1)

consists of finding a scalar function \( v \in L^2(0, T) \), called control, such that the corresponding solution \( y \) of \( (1) \) verifies

\[
y(T, \cdot) = 0.
\]

(2)
Several approaches are available for the study of a controllability problem:

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Methods to study the controllability

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- Optimization methods (Hilbert Uniqueness Method)
  - Multipliers
  - Carleman estimates
  - Microlocal Analysis

- Moment problem
- Construction and evaluation of biorthogonals
The null-controllability of the Schrödinger equation is equivalent to solve the following moment problem:

For any $y^0 = \sum_{n=1}^{\infty} a_n \sin(nx)$, find $v \in L^2(0,T)$ such that

$$\int_{-T/2}^{T/2} v \left( s + \frac{T}{2} \right) e^{s\nu_n} \, ds = \frac{(-1)^n \pi}{2ni} e^{-\frac{T}{2} \nu_n} a_n \quad \forall \, n \geq 1,$$

(3)

where $\nu_n = -i n^2$ are the eigenvalues of the unbounded skew-adjoint differential operator corresponding to (1).
Moment problem for the Schrödinger equation

For any $T > \frac{2\pi}{\gamma_\infty}$ the following inequality holds

**Ingham’s inequality**

$$\sum_{n \geq 1} |\alpha_n|^2 \leq C(T) \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n \geq 1} \alpha_n e^{\nu_n t} \right|^2 dt \right) \quad \forall (\alpha_n)_{n \geq 1} \in \ell^2. \quad (4)$$
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In our case (4) holds for any $T > 0$ due to the fact that

$$\gamma_\infty := \liminf_{n \to \infty} |\nu_{n+1} - \nu_n| = \infty.$$
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\sum_{n \geq 1} |\alpha_n|^2 \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n \geq 1} \alpha_n e^{\nu_n t} \right|^2 dt \quad \forall (\alpha_n)_{n \geq 1} \in \ell^2. \tag{4}
\]

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\]

Consequently, the moment problem has a solution $v \in L^2(0, T)$ for each $y^0 \in H^{-1}(0, \pi)$. This solution is not unique since the family of exponential functions $(e^{\nu_n t})_{n \geq 1}$ is not complete in $L^2(0, T)$. 
A solution $v$ of the moment problem may be constructed by means of a biorthogonal family to the sequence $(e^{\nu_n t})_{n \geq 1}$. 

\[ \int_{-T_2}^{T_2} \phi_m(t) e^{\nu_n t} \, dt = \delta_{mn} \quad \forall \, m, n \geq 1, \quad (5) \]

Once we have a biorthogonal sequence to $(e^{\nu_n t})_{n \geq 1}$, it follows that a "formal" solution of the moment problem is given by 

\[ v(t) = \sum_{n \geq 1} (-1)^n \pi a_n^2 \phi_n(t-T_2) e^{-T_2 \nu_n} \quad \forall \, t \in (0, T). \quad (6) \]
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**Definition**

A family of functions $(\phi_m)_{m \geq 1} \subset L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t)e^{\nu_n t} dt = \delta_{mn} \quad \forall m, n \geq 1,$$

is called a biorthogonal sequence to $(e^{\nu_n t})_{n \geq 1}$ in $L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)$. 

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$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{\bar{\nu}_n t} dt = \delta_{mn} \quad \forall m, n \geq 1,$$

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$$v(t) = \sum_{n \geq 1} \frac{(-1)^n \pi a_n}{2n i} \phi_n \left( t - \frac{T}{2} \right) e^{-\frac{T}{2} \bar{\nu}_n} \quad \forall t \in (0, T).$$

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\[
\hat{\phi}_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t)e^{-izt} dt \quad \text{(Fourier transform of } \phi_m). 
\]

\[
\phi_m \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right) \Rightarrow \left\{ \begin{array}{l}
\hat{\phi}_m \text{ is entire function of exponential type } \frac{T}{2} \\
\hat{\phi}_m \in L^2(\mathbb{R}) 
\end{array} \right. 
\]

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t)e^{\tilde{\nu}_n t} dt = \delta_{mn} \Rightarrow \hat{\phi}_m(i\tilde{\nu}_n) = \delta_{mn}, \quad n \geq 1.
\]
Moment problem for the Schrödinger equation

How to get a biorthogonal sequence to \((e^{\nu_n t})_{n \geq 1}\) in \(L^2\left(-\frac{T}{2}, \frac{T}{2}\right)\)?

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\(\hat{\phi}_m(z)\) with the above properties + Inverse Fourier Transform + Paley-Wiener Theorem \(\Rightarrow \phi_m.\)
Our aim is to study the possibility of obtaining a control for (1) as limit of controls of the following perturbed equations

\[
\begin{aligned}
    y_t(t, x) - i y_{xx}(t, x) &= 0, & x \in (0, \pi), & t > 0 \\
    y(t, 0) &= 0, & y(t, \pi) &= v(t), & t > 0 \\
    y(0, x) &= y^0(x), & x \in (0, \pi), 
\end{aligned}
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\begin{cases}
    y_t(t, x) - i y_{xx}(t, x) - \varepsilon y_{xx}(t, x) = 0, & x \in (0, \pi), \ t > 0 \\
    y(t, 0) = 0, \ y(t, \pi) = v_\varepsilon(t), & t > 0 \\
    y(0, x) = y^0(x), & x \in (0, \pi),
\end{cases}
\]

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where \( \varepsilon \) is a small parameter devoted to tend to zero.
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**Definition**

A function \( v_{\varepsilon} \in L^{2}(0, T) \) is called control for the initial datum \( y^{0} \) in time \( T \) if the corresponding solution of (7) verifies \( y(T, \cdot) = 0 \). If, given any \( y^{0} \in H^{-1}(0, \pi) \), there exists a control \( v_{\varepsilon} \in L^{2}(0, T) \) for \( y^{0} \) in time \( T \), we say that (7) is null-controllable in time \( T \).
We are interested in the following issue:

Given $T > 0$, $\varepsilon > 0$ and $y^0 \in H^{-1}(0, \pi)$, is there a control $v_\varepsilon \in L^2(0, T)$ for (7) such that the family $(v_\varepsilon)_{\varepsilon > 0}$ converges to a control of (1) when $\varepsilon \to 0$?
We are interested in the following issue:

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In (7), $-\varepsilon y_{xx}$ represents a viscous term. Indeed, if $v_\varepsilon = 0$,

$$\frac{d}{dt} \int_0^\pi |y(t, x)|^2 dx = -2 \varepsilon \int_0^\pi |y_x(t, x)|^2 dx \leq 0 \quad \forall t \geq 0,$$

and the $L^2$-norm of the solution $y$ of (7) is decreasing in time.
Controllability of the perturbed Schrödinger equation

- (7) may be viewed as a Schrödinger equation with an added viscous term. As such it models an array of optical fibers in a weakly lossy medium.


- (7) is known as the linear complex Ginzburg-Landau equation. With a cubic nonlinearity, it plays an important role in the theory of amplitude equations and provides a simple model for turbulence.

The interest of this problem is justified by the use of the vanishing viscosity as a typical mechanism to

- **Study Cauchy problems**
  

- **Improve convergence of numerical schemes**
  

In both examples the viscosity is devoted to tend to zero in order to obtain the original system. Thus, a legitimate question is related to the behavior and the sensitivity of the controls during this process.
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In [1] the authors pass to the limit in the dissipative wave equation to obtain a control for the heat equation.

In [2,3] the controllability of the transport equation is considered by introducing a vanishing viscosity term. In [2] Carleman estimates are used to obtain a uniform bound for the family of controls. The same result is shown in [3], improving the control time, by means of nonharmonic Fourier analysis and biorthogonal technique.

The recent article [4] deals with a nonlinear scalar conservation law perturbed by a small viscosity term and proves the uniform boundedness of the boundary controls.
Theorem

There exists $T > 0$ with the property that, for any $y_0 \in H^{-1}(0, \pi)$ and $\varepsilon \in (0, 1]$, there exists a control $v_\varepsilon \in L^2(0, T)$ of (7) such that the family $(v_\varepsilon)_{\varepsilon \in (0, 1]}$ is uniformly bounded in $L^2(0, T)$ and any weak limit $v$ of it is a control in time $T$ for the Schrödinger equation (1).

Uniform controllability in time $T$ independent of $\varepsilon$.

$y_0 \in H^{-1}(0, \pi)$ is the optimal space.

The same result should be true for arbitrary $T > 0$ (optimal time).
Main result

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- Uniform controllability in time $T$ independent of $\varepsilon$.
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- Uniform controllability in time \( T \) independent of \( \varepsilon \).
- \( y^0 \in H^{-1}(0, \pi) \) is the optimal space.
- The same result should be true for arbitrary \( T > 0 \) (optimal time).
Given \( y^0 = \sum_{n=1}^{\infty} a_n \sin(nx) \), its null-controllability is equivalent to solve the moment problem:

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} v_\varepsilon \left( s + \frac{T}{2} \right) e^{s\mu_n} ds = \frac{(-1)^n \pi}{2n(i + \varepsilon)} e^{-\frac{T}{2} \mu_n} a_n \quad \forall \ n \geq 1, \tag{9}
\]

where \( \mu_n = \varepsilon n^2 - i n^2 \) are the eigenvalues of the associated differential operator.
Proof of the main theorem: Step 1 - Moment problem

Given $y^0 = \sum_{n=1}^{\infty} a_n \sin(nx)$, its null-controllability is equivalent to solve the moment problem:

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where $\mu_n = \epsilon n^2 - i n^2$ are the eigenvalues of the associated differential operator.

- The exponents $\mu_n$ are no longer purely imaginary. We cannot use Ingham’s inequality.
- Can we apply the technique of Fattorini H. O. and Russell D. L.? 
A solution $v$ of the moment problem may be constructed by means of a biorthogonal family to the sequence $(e^{\mu_n t})_{n \geq 1}$. 

Definition

A family of functions $(\zeta_m)_{m \geq 1} \subset L^2(-T_2, T_2)$ with the property

$$\int_{-T_2}^{T_2} \zeta_m(t) e^{\mu_n t} dt = \delta_{mn} \quad \forall m,n \geq 1,$$

(10)

is called a biorthogonal sequence to $(e^{\mu_n t})_{n \geq 1}$ in $L^2(-T_2, T_2)$.

The element $\zeta_m$ of the biorthogonal sequence controls the one mode initial datum $y_0^m(x) = \sin(mx)$.

If we have a finite combination of modes that we want to control, we consider a finite combination of biorthogonals.
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- The element $\zeta_m$ of the biorthogonal sequence controls the one mode initial datum $y_m^0(x) = \sin(mx)$.
- If we have a finite combination of modes that we want to control, we consider a finite combination of biorthogonals.
If we have an initial datum with an infinity of modes, a “formal” solution of the moment problem is given by

\[ v_\varepsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \pi a_n}{2n(i + \varepsilon)} e^{-\frac{T}{2} \mu_n} \zeta_n \left( t - \frac{T}{2} \right) \quad \forall t \in (0, T). \] (11)
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The absolute convergence of this series depends on

- The Fourier coefficients \( a_n \) of the controlled initial datum \( y^0 \).
- The norm of the biorthogonals elements \( \zeta_n \).
- The exponentials \( e^{-\frac{T}{2} \mu_n} \) “help” the convergence in the range \( n \geq \frac{1}{\sqrt{\varepsilon}} \),

\[ \left| e^{-\frac{T}{2} \mu_n} \right| = e^{-\frac{T}{2} \varepsilon n^2}. \]
Proof of the main theorem: Step 2 - Biorthogonals

If we have an initial datum with an infinity of modes, a “formal” solution of the moment problem is given by

\[ v_\varepsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \pi a_n}{2n(i + \varepsilon)} e^{-\frac{T}{2} \mu_n \zeta_n} \left( t - \frac{T}{2} \right) \quad \forall t \in (0, T). \]  \hspace{1cm} (11)

To prove the existence of a control \( v_\varepsilon \) for any \( y^0 \in H^{-1}(0, \pi) \) and to get uniform bounds for its \( L^2 \)-norm we need:

- Existence of biorthogonals
- Evaluation of their norms
  - uniformly bounded in the range \( n < \frac{1}{\sqrt{\varepsilon}} \).
  - not too big in the range \( n \geq \frac{1}{\sqrt{\varepsilon}} \).
The existence of a biorthogonal family to \((e^{\mu_n t})_{n \geq 1}\) in \(L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)\) follows from the lack of completeness of the family \((e^{\mu_n t})_{n \geq 1}\) in \(L^2 \left(-\frac{T}{2}, \frac{T}{2}\right)\).

**Theorem (Müntz-Szász, 1916)**

Let \((\lambda_n)_{n \geq 1}\) be a sequence of complex numbers such that \(\liminf_{n \to \infty} \mathcal{R}(\lambda_n) > 0\) and \(T > 0\). Then the set of functions \((e^{-\lambda_n t})_{n \geq 1}\) is complete in \(L^2 (0, T)\) if and only if

\[
\sum_{n=1}^{\infty} \frac{\mathcal{R}(\lambda_n)}{1 + |\lambda_n|^2} = \infty. \tag{12}
\]

In our case: 

\[
\sum_{n=1}^{\infty} \frac{\mathcal{R}(\mu_n)}{1 + |\mu_n|^2} = \sum_{n=1}^{\infty} \frac{\varepsilon n^2}{1+n^4+\varepsilon^2 n^4} < \infty.
\]
The evaluation of the norm of a biorthogonal family to \((e^{\mu_n t})_{n \geq 1}\) in \(L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)\) is obtained from an explicit construction.

**Theorem**

There exists \(T > 0\) such that, for any \(\varepsilon \in (0, 1]\), we find a biorthogonal sequence \((\zeta_m)_{m \in \mathbb{N}^*}\) to the family \((e^{\mu_m t})_{m \in \mathbb{N}^*}\) in \(L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)\) with the following property

\[
\|\zeta_m\|_{L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)} \leq C \exp(\alpha |\Re(\mu_m)|) \quad \forall m \geq 1, \tag{13}
\]

where \(C\) and \(\alpha\) are positive constants independent of \(\varepsilon\) and \(m\).
Proof: For any $m \in \mathbb{N}^*$, we define the function

$$P_m(z) = \prod_{\substack{n \in \mathbb{Z}^* \mid |n| \neq m}} \left(1 - \frac{z}{i \lambda_n}\right) \left(\frac{\lambda_n}{\lambda_n - \lambda_m}\right), \quad (14)$$

where

$$\lambda_n = \begin{cases} 
i n + \varepsilon n, & \text{if } n = q^2 \quad \text{with } q \in \mathbb{N}^* \\
i \sqrt{1 + \varepsilon^2} n, & \text{if } n \neq q^2, n > 0 \quad \text{with } q \in \mathbb{N}^* \\
\lambda_{-n}, & \text{if } n < 0. \end{cases}$$

Note that $\lambda_{p^2} = \overline{\mu_p}$, for any $p \geq 1$. Hence, the family $(\lambda_m)_{m \geq 1}$ is “larger” than $(\mu_p)_{p \geq 1}$. This extension of the family of exponents will be very important for the behavior on the real axis of $P_m$. 
Proof of the main theorem: Step 4 - Evaluation

**Lemma**

For each \( m \geq 1 \) of the form \( m = p^2 \) with \( p \in \mathbb{N}^* \), \( P_m \) is an entire function of exponential type independent of \( \varepsilon \) such that

\[
P_m(i\lambda_n) = \delta_{mn}, \quad n \in \mathbb{N}^*.
\]  

(15)
Lemma

For each $m \geq 1$ of the form $m = p^2$ with $p \in \mathbb{N}^*$, $P_m$ is an entire function of exponential type independent of $\varepsilon$ such that

$$P_m(i\lambda_n) = \delta_{mn}, \quad n \in \mathbb{N}^*. \quad (15)$$

Lemma

For each $m \geq 1$ of the form $m = p^2$ with $p \in \mathbb{N}^*$, the function $P_m$ defined by (14) has the following property

$$|P_m(x)| \leq C \exp(32\varepsilon \sqrt{|x|} + 16\varepsilon \Re(\lambda_m)) \quad \forall x \in \mathbb{R}, \quad (16)$$

where $C$ is a positive constant, independent of $\varepsilon$ and $m$. 
Lemma

For any $\varepsilon \in (0, 1]$ and $m \geq 1$ of the form $m = p^2$ with $p \in \mathbb{N}^*$, there exists a function $M_{m,\varepsilon} : \mathbb{C} \to \mathbb{C}$ such that

1. $M_{m,\varepsilon}$ is an entire function of exponential type independent of $\varepsilon$ and $m$.
2. $|M_{m,\varepsilon}(x)| \leq \exp \left( -\varepsilon \sqrt{|x|} \right)$ for all $x \in \mathbb{R}$.
3. $|M_{m,\varepsilon}(i\lambda_m)| \geq C \exp \left( -R |\Re(\lambda_m)| \right)$,

where $C$ and $R$ are positive constants independent of $\varepsilon$ and $m$. 
For each $m \in \mathbb{N}^*$ we define the function

$$
\Psi_m(z) = P_{m^2}(z) \left( \frac{M_{m^2,\varepsilon}(z)}{M_{m^2,\varepsilon}(i\lambda_{m^2})} \right)^{32} \frac{\sin(\gamma(z - i\lambda_{m^2}))}{\gamma(z - i\lambda_{m^2})} \quad \forall z \in \mathbb{C},
$$

where $\gamma > 0$ is an arbitrary constant independent of $\varepsilon$ and $m$. 
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where \( \gamma > 0 \) is an arbitrary constant independent of \( \varepsilon \) and \( m \).

- \( \Psi_m \) is an entire function of exponential type \( \frac{T}{2} > 0 \).
- \( \Psi_m \in L^2(\mathbb{R}), \|\Psi_m\|_{L^2(\mathbb{R})} \leq C \exp(\alpha|\Re(\mu_m)|) \).
- \( \Psi_m(i\bar{\mu}_n) = \delta_{mn} \).
From Paley-Wiener’s Theorem we deduce that there exists $(\zeta_m)_{m \geq 1} \subset L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)$ such that

$$\Psi_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta_m(t) e^{-izt} \, dx \quad \forall m \geq 1.$$ 

- $(\zeta_m)_{m \geq 1} \subset L^2 \left( -\frac{T}{2}, \frac{T}{2} \right)$ is a biorthogonal sequence.

- $\|\zeta_m\|_{L^2\left( -\frac{T}{2}, \frac{T}{2} \right)} = 2\pi \|\Psi_m\|_{L^2(\mathbb{R})} \leq C \exp \left( \alpha |\Re(\mu_m)| \right).$

End of the proof.
The simplest function $P_m$ does not work. Indeed, if we choose

$$P_m(z) = \prod_{\substack{n \in \mathbb{Z}^* \\ |n| \neq m}} \left( 1 - \frac{z}{i \mu_n} \right) \left( \frac{\mu_n}{\mu_n - \bar{\mu}_m} \right),$$

(17)

we have an entire function which verifies the necessary relations $P_m(i \bar{\mu}_n) = \delta_{mn}$. Moreover, a product like (17), has arbitrarily small exponential type and eventually would allow to deduce a controllability result for any $T > 0$. 
The simplest function $P_m$ does not work. Indeed, if we choose

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we have an entire function which verifies the necessary relations $P_m(i \bar{\mu}_n) = \delta_{mn}$. Moreover, a product like (17), has arbitrarily small exponential type and eventually would allow to deduce a controllability result for any $T > 0$.

However, $P_m$ is useless. Indeed,

$$|P_m(x)| \geq C_1 \exp \left(C_2 \sqrt{|x|}\right) \quad \forall x \in \mathbb{R}$$

with $C_1$ and $C_2$ two positive constants independent of $\varepsilon$. 
For each $\varepsilon \in (0, 1]$ and $m \geq 1$, we need a multiplier $M_{m, \varepsilon}$,

$$|M_{m, \varepsilon}(x)| \leq \exp \left( -C_2 \sqrt{|x|} \right) \quad \forall x \in \mathbb{R},$$  

(18)

$$|M_{m, \varepsilon}(i\mu_m)| \geq \exp \left( -C_3 |\Re(\mu_m)| \right) = \exp \left( -C_3 \varepsilon m^2 \right),$$  

(19)

with $C_3$ a positive constant independent of $\varepsilon$. 


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\]

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with \( C_3 \) a positive constant independent of \( \varepsilon \).

\[
G_{m, \varepsilon}(z) = M_{m, \varepsilon}(z) e^{(B+\eta)z} e^{C_2 \sqrt{-z}}
\]

From the Phragmén-Lindelöf Theorem, there exists a constant \( C > 0 \), independent of \( \varepsilon \) and \( m \), such that

\[
|G_{m, \varepsilon}(x + iy)| \leq C, \quad x \leq 0, \; y \geq 0.
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This gives the contradiction:

$$|M_{m, \varepsilon}(i \mu_m)| \leq C \exp \left( -\frac{m}{\sqrt{2}} \right), \quad m \geq 1.$$  

(20)
For any finite sequence \((a_n)_{n \geq 1}\), the following weighted Ingham type inequality holds

\[
\sum_{n \geq 1} |a_n|^2 e^{-2\alpha |\Re(\mu_n)|} \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n \geq 1} a_n e^{\mu_n t} \right|^2 dt.
\]  (21)
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Other types of control problem can be addressed: punctual controls \(\delta_{x_0}v(t)\) or lumped controls \(v(t)g(x)\).
For any finite sequence \((a_n)_{n \geq 1}\), the following weighted Ingham type inequality holds

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\]

Other types of control problem can be addressed: punctual controls \(\delta_{x_0} v(t)\) or lumped controls \(v(t)g(x)\).

Can we extend the result to several dimensions? Probably yes, by using Carleman estimates (work in progress).
Related problems

\[
\begin{cases}
    u_j''(t) - \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{h^2} - \varepsilon \frac{u_{j+1}'(t) + u_{j-1}'(t) - 2u_j'(t)}{h^2} = 0 \\
    u_0(t) = 0, \quad u_{N+1}(t) = v_h(t), \\
    u_j(0) = u_j^0(x), \quad u_j' = u_j^1(x).
\end{cases}
\]

(22)

If \( \varepsilon = h \) there exists a sequence of boundary controls for (22), \((v_h)_h \rangle > 0\) which is uniformly bounded in \( L^2(0,T) \) and any weak limit of it is a boundary control of the continuous wave equation.


If \( \varepsilon = h^2 \) this is no longer true.

What is happening when \( \varepsilon = h^\alpha \) with \( \alpha \in (1,2) \)?
Related problems

\[
\begin{aligned}
\left\{
\begin{array}{l}
 u''_j(t) - \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{h^2} - \frac{\varepsilon u'_{j+1}(t) + u'_{j-1}(t) - 2u'_j(t)}{h^2} = 0 \\
 u_0(t) = 0, \quad u_{N+1}(t) = v_h(t), \\
 u_j(0) = u^0_j(x), \quad u'_j = u^1_j(x).
\end{array}
\right.
\end{aligned}
\]

(22)

- If \( \varepsilon = h \) there exists a sequence of boundary controls for (22), \((v_h)_h>0\) which is uniformly bounded in \( L^2(0, T) \) and any weak limit of it is a boundary control of the continuous wave equation.


- If \( \varepsilon = h^2 \) this is no longer true.

- What is happening when \( \varepsilon = h^\alpha \) with \( \alpha \in (1, 2) \)?
Related problems

\[
\begin{cases}
    u_t + i(-\Delta)^{\frac{1}{2}}u - \varepsilon \Delta^{\alpha}u = g(x)v_{\varepsilon}(t), & x \in (0, \pi), \ t > 0 \\
    u(t, 0) = u(t, \pi) = 0, & t > 0 \\
    u(0, x) = u_0(x), & x \in (0, \pi),
\end{cases}
\]

(23)
\begin{equation}
\begin{cases}
    u_t + i(-\Delta)^{\frac{1}{2}} u - \varepsilon \Delta^\alpha u = g(x)v_\varepsilon(t), \quad x \in (0, \pi), \ t > 0 \\
    u(t, 0) = u(t, \pi) = 0, \quad t > 0 \\
    u(0, x) = u_0(x), \quad x \in (0, \pi),
\end{cases}
\end{equation}

- \alpha = 1/2: We do not have spectral controllability: Müntz-Szász condition is not satisfied.

- \alpha = 1: The dissipation is very strong and we have uniform controllability.


- \alpha \in (1/2, 1)? Müntz-Szász condition is still satisfied. Do we still have uniform controllability?
THANK YOU!