Superconvergent functional estimates from summation-by-parts finite difference discretizations

A paper by Jason E. Hicken and David W. Zingg

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Outline

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- SBP operators mimic the stability of Galerkin finite-element methods.
- SAT penalties offer time-stability.
- SBP-SAT discretizations achieve both stability and high-order accuracy, but on boundaries that accuracy is decreased.
- Discretizations based on diagonal-norm SBP-SAT operators produce superconvergent functional estimates.

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Notation

- Uniformly spaced mesh grid on $[0,1]$:

$$x_k = kh, \text{ with } k = 0, 1, ..., n \text{ for } h = 1/n$$

- $\mathcal{U} \in \mathcal{C}^p([0,1])$
- $u = \begin{pmatrix} \mathcal{U}(x_0) & \mathcal{U}(x_1) & \dots & \mathcal{U}(x_n) \end{pmatrix}^T \in \mathbb{R}^{n+1}$
- $u_h \in \mathbb{R}^{n+1}$
- $e_0 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix}^T \in \mathbb{R}^{n+1}$
- $e_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}^T \in \mathbb{R}^{n+1}$
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J. E. Hicken and D. W. Zingg, *Summation-by-parts operators and high-order quadrature*, http://arxiv.org/abs/1003.5182

Summation-by-parts operators

SBP operator

The matrix $D \in \mathbb{R}^{(n+1)\times(n+1)}$ is a summation-by-parts operator for the first derivative if it has the form

$$D = H^{-1}Q$$

where $H \in \mathbb{R}^{(n+1)\times(n+1)}$ is a symmetric-positive-definite weight matrix with entries $H_{ij} = \mathcal{O}(h)$ and $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfies

$$Q + Q^T = \operatorname{diag}(-1, 0, \ldots, 0, 1) = e_n e_n^T - e_0 e_0^T$$

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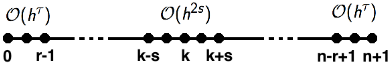
Remark: D is a $2s$-order approximation to d/dx at the $n+1-2r$ interior nodes and a τ-order accurate approximation at the $r+r$ boundary nodes.



Summation-by-parts operators

We will consider SBP operators such that:

$$H = h \begin{pmatrix} H_L & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & H_R \end{pmatrix}$$

where $H_L = \text{diag}(\rho_0, \rho_1, \ldots, \rho_{r-1})$ and $H_R = \text{diag}(\rho_{r-1}, \ldots, \rho_1, \rho_0)$.

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Remember:
Due to the truncation error at the boundary, the solution error is $\mathcal{O}(h^{s+1})$.

B. Gustaffson, *The convergence rate for difference approximations to general mixed initial boundary value problems*, SIAM Journal of Numerical Analysis, vol. 18, 1992, pp. 179-190.

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- It is symmetric and positive-definite.
- It defines an inner product and a norm for vectors in the grid:

$$(u, z)_H := u^T H z \qquad \|u\|_H^2 := (u, u)_H$$

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- Functional estimates based on these discretizations are $2s$-accurate.

Summation-by-parts operators

Lemma

Let $D = H^{-1}Q$ be an SBP first derivative operator. Then

$$(z, Du)_H = z^T Qu$$

is a $2s$-order accurate approximation to the integral

$$\int_0^1 \mathcal{Z} \frac{d\mathcal{U}}{dx} dx$$

where $\mathcal{Z}\frac{d\mathcal{U}}{dt} \in \mathcal{C}^{2s-1}([0,1])$.

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Simultaneous approximation term

Let us consider:

$$\begin{cases} \dfrac{\partial \mathcal{U}}{\partial t} + a\dfrac{\partial \mathcal{U}}{\partial x} = 0, \ \ x \in \Omega = [0,1] \ (a > 0) \\ \mathcal{U}(x,0) = \mathcal{U}_0(x) \\ \mathcal{U}(0,t) = \mathcal{U}_L(t) \end{cases}$$

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We can impose boundary conditions in a strong sense. For example, by the injection method:

$$\left[I - e_0 e_0^T\right]\left[\frac{\partial u_h}{\partial t} + aDu_h\right] + e_0\left[e_0^T u_h - \mathcal{U}_L(t)\right] = 0$$

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It can destroy stability properties

K. Mattsson, *Boundary procedures for summation-by-parts operators*, Journal on Scientific Computing, vol. 18, 2003, pp. 133-153.

Simultaneous approximation term

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A SAT is a penalty term used to impose the boundary conditions in a weak sense.

- M. H. Carpenter, D. Gottlieb and S. Abarbanel, *Time-stable boundary conditions for finite difference schemes solving hyperbolic systems: Methodology and applications to high-order compact schemes*, Journal of Computational Physics, vol. 111, 1994, pp. 220-236.
- D. Funaro and D. Gottlieb, *A new method imposing boundary conditions in pseudospectral approximations of hyperbolic equations*, Mathematics of Computation, vol. 51, 1988, pp. 599-613.

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We now obtain the semidiscretization (with SAT):

$$\frac{\partial u_h}{\partial t} + aDu_h = -\sigma a H^{-1} e_0 \left[e_0^T u_h - \mathcal{U}_L(t) \right]$$

where σ is a penalty parameter.

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Time-stability

Using the SAT, the semidiscretization is time-stable; i.e., for $\mathcal{U}_L = 0$ it holds $\|u_h(t)\| \leq C\|u_0\|$ for a certain $C \in \mathbb{R}$.

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$$\frac{d}{dt}\|u_h\|_H^2 = -au_h^T(Q+Q^T)u_h - 2\sigma a u_0^2 \qquad (Q+Q^T = e_n e_n^T - e_0 e_0^T)$$

$$= -au_n^2 + a(1-2\sigma)u_0^2 \leq 0 \Leftrightarrow \sigma \geq \frac{1}{2}$$

Duality

Let us consider a linear functional $\mathcal{I}(\mathcal{U}) = (\mathcal{G}, \mathcal{U})_\Omega$ where $\mathcal{U}$ satisfies $\mathcal{L}\mathcal{U} - \mathcal{F} = 0$ for a linear differential operator $\mathcal{L}$ and $\mathcal{F} \in L_2(\Omega)$. Then, for a generic function $\mathcal{V}$:

$$\mathcal{I}(\mathcal{U}) = (\mathcal{V}, \mathcal{F})_\Omega - (\mathcal{L}^*\mathcal{V} - \mathcal{G}, \mathcal{U})_\Omega$$

where $\mathcal{L}^*$ is such that $(\mathcal{V}, \mathcal{L}\mathcal{U})_\Omega = (\mathcal{L}^*\mathcal{V}, \mathcal{U})_\Omega$. The associated adjoint problem is:

$$\mathcal{L}^*\mathcal{V} - \mathcal{G} = 0, \ \forall x \in \Omega$$

Duality

Let us consider a linear functional $\mathcal{I}_h(u_h) = (g, u_h)_h$ where u_h satisfies the associated discretization $\mathcal{L}_h u_h - f = 0$. Then, for a generic vector $v_h \in \mathbb{R}^{n+1}$:

$$\mathcal{I}_h(u_h) = (v_h, f)_h - (\mathcal{L}_h^T v_h - g, u_h)_h$$

where $\mathcal{L}_h^T$ is such that $(v_h, \mathcal{L}_h u_h)_h = (\mathcal{L}_h^T v_h, u_h)_h$. The discrete adjoint problem is:

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Dual-consistency

A discrete operator $\mathcal{L}_h$ and functional $\mathcal{J}_h$ are dual-consistent of order $q \geq 1$ with respect to a corresponding continuous PDE and functional if

$$\mathcal{L}_h^T v - g = \mathcal{O}(h^q)$$

where v is the solution to the continuous adjoint problem projected onto the discrete solution space.

Outline

Linear hyperbolic PDEs

Let us consider a scalar hyperbolic PDE:

$$\begin{cases} \dfrac{\partial \mathcal{U}}{\partial t} + \dfrac{\partial}{\partial x}(\lambda \mathcal{U}) = \mathcal{F}, \ x \in \Omega = [0,1], \ t > 0 \\ \mathcal{U}(x,0) = \mathcal{U}_0(x) \\ \mathcal{U}(0,t) = \mathcal{U}_L(t) \end{cases}$$

where $\lambda(x)$ is the spatially varying wave speed.

Linear hyperbolic PDEs

Let us consider a stationary scalar hyperbolic PDE:

$$\begin{cases} \dfrac{\partial}{\partial x}(\lambda \mathcal{U}) = \mathcal{F}, \ x \in \Omega = [0, 1] \\ \mathcal{U}(0) = \mathcal{U}_L \end{cases}$$

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where $\lambda(x)$ is the spatially varying wave speed. The SBP-SAT discretization is given by:

$$D(\Lambda u_h) = f - H^{-1} e_0 \lambda_0 (e_0^T u_h - \mathcal{U}_L) \qquad (\star)$$

where

$$\begin{aligned} \lambda_i &= \lambda(x_i), \;\; i = 0, ..., n \\ \Lambda &= diag(\lambda_0, ..., \lambda_n) \\ f &= \begin{bmatrix} \mathcal{F}(x_0) & \ldots & \mathcal{F}(x_n) \end{bmatrix}^T \end{aligned}$$

Linear hyperbolic PDEs

Lemma

Diagonal SBP operators with s-accurate boundary closure, the solution error $\|u_h - u\|_p$ is at best $\mathcal{O}(h^{s+1})$.

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Lemma

Diagonal SBP operators with s-accurate boundary closure, the solution error $\|u_h - u\|_p$ is at best $\mathcal{O}(h^{s+1})$.

Theorem

Let $\mathcal{I} : L^2(\Omega) \longrightarrow \mathbb{R}$ be a linear functional defined by:

$$\mathcal{I}(\mathcal{U}) = \int_0^1 \mathcal{G}\mathcal{U}\,dx + \alpha(\lambda\mathcal{U})|_{x=1}$$

where $\mathcal{G} \in \mathcal{C}^{2s}([0,1])$ and let $H^{-1}A := H^{-1}(Q + e_0 e_0^T)\Lambda$ denote the matrix appearing in the linear system $(\star)$. Assume that the PDE is well posed and admits solution $\mathcal{U} \in \mathcal{C}^{2s}([0,1])$ and $(\star)$ is non-singular. Then, if $\|A^{-T}H\|_\infty \leq C$ for some constant $C \in \mathbb{R}$ that does not depend on n, the functional estimate:

$$\mathcal{I}_h(u_h) = (g, u_h)_H + \alpha\lambda_n(e_n^T u_h)$$

where $g = [\mathcal{G}(x_0) \ \ \dots \ \ \mathcal{G}(x_n)]^T$, is a $2s$-order accurate approximation to $\mathcal{I}(\mathcal{U})$.

Linear hyperbolic PDEs

Proof:

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Using the SBP norm:

$$\begin{aligned}
\mathcal{I}(\mathcal{U}) &= (g,u)_H + \alpha\lambda_n(e_n^T u) + \mathcal{O}(h^{2s}) \\
&= g^T H u + \alpha\lambda_n e_n^T u + \mathcal{O}(h^{2s}) \\
&= g^T H u_h - g^T H(u_h - u) + \alpha\lambda_n e_n^T u_h - \alpha\lambda_n e_n^T(u_h - u) + \mathcal{O}(h^{2s}) \\
&= \mathcal{I}_h(u_h) - (g^T + \alpha\lambda_n e_n^T H^{-1}) H (u_h - u) + \mathcal{O}(h^{2s})
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Left-multiplying $(\star)$ by H:

$$Au_h = Hf + \lambda_0 e_0 \mathcal{U}_L$$

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In the case that the equation above is applied to the exact solution u:

$$Au = HD(\Lambda u) + \lambda_0 e_0 \mathcal{U}_L$$

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Using both expressions and $I = HA^{-1}AH^{-1}$, we obtain:

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\mathcal{I}(\mathcal{U})) &= \mathcal{I}_h(u_h) - (g^T + \alpha\lambda_n e_n^T H^{-1})HA^{-1}AH^{-1}H(u_h - u) + \mathcal{O}(h^{2s}) \\
&= \mathcal{I}_h(u_h) - (g^T H + \alpha\lambda_n e_n^T)A^{-1}H(f - D\Lambda u) + \mathcal{O}(h^{2s}) \\
&= \mathcal{I}_h(u_h) - v_h^T H(f - D\Lambda u) + \mathcal{O}(h^{2s})
\end{aligned}$$

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$$-\Lambda D v_h = g - H^{-1} e_n \lambda_n (e_n^T v_h - \alpha)$$

which is, exactly, the SBP-SAT discretization of the corresponding adjoint problem of our PDE:

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$$\begin{cases} -\lambda\dfrac{\partial\mathcal{V}}{\partial x} = \mathcal{G}, \ x \in \Omega = [0,1] \\ \mathcal{V}(1) = \alpha \end{cases}$$

The truncation error is given by $T_h = g + H^{-1}e_n\lambda_n\alpha - H^{-1}A^T v$ and, because it is a SBP discretization, $\|T_h\|_\infty = \mathcal{O}(h^s)$. Therefore, $H^{-1}A^T(v_h - v) = T_h$ and, using the hypothesis on $\|A^{-T}H\|_\infty$, we obtain:

$$\|v_h - v\|_\infty \leq C\|T_h\|_\infty = \mathcal{O}(h^s)$$

Linear hyperbolic PDEs

Proof (end):
Finally:

$$\begin{aligned} \mathcal{I}(\mathcal{U}) &= \mathcal{I}_h(u_h) - v_h^T H(f - D\Lambda u) + \mathcal{O}(h^{2s}) \\ &= \mathcal{I}_h(u_h) - v^T H(f - D\Lambda u) + (v_h - v)^T H(f - D\Lambda u) + \mathcal{O}(h^{2s}) \end{aligned}$$

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\end{aligned}$$

■

Remark: The SBP-SAT discretization of the PDE and the discrete functional estimate lead to a discrete adjoint problem that is a consistent and sufficiently accurate discretization of the adjoint PDE; i. e., it is dual-consistent.

Linear elliptic PDEs

Let us consider a linear elliptic PDE, expressed in the form of first-order system:

$$\begin{cases} -\dfrac{\partial}{\partial x}(\gamma \mathcal{U}) = \mathcal{F}, \ x \in \Omega = [0,1] \\ \mathcal{W} = \dfrac{\partial \mathcal{U}}{\partial x}, \ x \in \Omega \\ \mathcal{U}(0) = \mathcal{U}_L \\ \mathcal{W}(1) = \mathcal{W}_R \end{cases}$$

where $\gamma(x) > 0$ is the spatially varying diffusion coefficient.

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where $\gamma(x) > 0$ is the spatially varying diffusion coefficient. The SBP-SAT discretization is given by:

$$\begin{aligned} -D(\Gamma w_h) = f - H^{-1} e_0 (e_0^T u_h - \mathcal{U}_L) - H^{-1} e_n \gamma_n (e_n^T w_h - \mathcal{W}_R) \\ w_h = D u_h + H^{-1} e_0 (e_0^T u_h - \mathcal{U}_L) \end{aligned} \qquad (\star\star)$$

where

$$\begin{aligned} \gamma_i &= \gamma(x_i), \;\; i = 0, ..., n \\ \Gamma &= diag(\gamma_0, ..., \gamma_n) \\ f &= \begin{bmatrix} \mathcal{F}(x_0) & \dots & \mathcal{F}(x_n) \end{bmatrix}^T \end{aligned}$$

Linear elliptic PDEs

Theorem

Let $\mathcal{I} : L^2(\Omega) \longrightarrow \mathbb{R}$ be a linear functional defined by:

$$\mathcal{I}(\mathcal{U}) = \int_0^1 \mathcal{G}_1 \mathcal{U} dx + \int_0^1 \mathcal{G}_2 \frac{\partial \mathcal{U}}{\partial x} dx + \alpha(\mathcal{U})|_{x=1} + \beta(\gamma \frac{\partial \mathcal{U}}{\partial x})\Big|_{x=0}$$

where $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{C}^{2s}([0,1])$. Assume that the PDE is well posed and admits solution $\mathcal{U} \in \mathcal{C}^{2s}([0,1])$. Then, the functional estimate:

$$\mathcal{I}_h(u_h) = (g_1, u_h)_H + (g_2, w_h)_H + \alpha(e_n^T u_h) + \beta\gamma_0(e_0^T w_h) + \beta(e_0^T u_h - \mathcal{U}_L)$$

where $g_k = [\mathcal{G}_k(x_0) \quad \dots \quad \mathcal{G}_k(x_n)]^T$ (with $k = 1, 2$), is a $2s$-order accurate approximation to $\mathcal{I}(\mathcal{U})$.

Linear elliptic PDEs

Proof:

Linear elliptic PDEs

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Using the SBP norm:

$$
\begin{aligned}
\mathcal{I}(\mathcal{U}) &= (g_1, u)_H + (g_2, w)_H + \alpha(e_n^T u) + \beta\gamma_0(e_0^T w) + \mathcal{O}(h^{2s}) \\
&= g_1^T H u_h + g_2^T H w_h + \alpha e_n^T u_h + \beta\gamma_0(e_0^T w_h) + \beta e_0^T(u_h - u) \\
&\quad - g_1^T H(u_h - u) - g_2^T H(w_h - w) - \alpha e_n^T(u_h - u) \\
&\quad - \beta\gamma_0 e_0^T(w_h - w) - \beta e_0^T(u_h - u) + \mathcal{O}(h^{2s}) \\
&= \mathcal{I}_h(u_h) - (g_1^T + \alpha e_n^T H^{-1} + \beta e_0^T H^{-1})H(u_h - u) \\
&\quad - (g_2^T + \beta\gamma_0 e_0^T H^{-1})H(w_h - w) + \mathcal{O}(h^{2s})
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Linear elliptic PDEs

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&= g_1^T H u_h + g_2^T H w_h + \alpha e_n^T u_h + \beta\gamma_0(e_0^T w_h) + \beta e_0^T (u_h - u) \\
&\quad - g_1^T H(u_h - u) - g_2^T H(w_h - w) - \alpha e_n^T (u_h - u) \\
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\end{aligned}$$

We define the primal-equation residuals as:

$$\begin{aligned}
r_u &:= D(\Gamma w_h) + f - H^{-1} e_0(e_0^T u_h - \mathcal{U}_L) - H^{-1} e_n \gamma_n (e_n^T w_h - \mathcal{W}_R) \\
r_w &:= D u_h - w_h + H^{-1} e_0(e_0^T u_h - \mathcal{U}_L)
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Linear elliptic PDEs

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&= g_1^T H u_h + g_2^T H w_h + \alpha e_n^T u_h + \beta\gamma_0(e_0^T w_h) + \beta e_0^T (u_h - u) \\
&\quad - g_1^T H(u_h - u) - g_2^T H(w_h - w) - \alpha e_n^T (u_h - u) \\
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&= \mathcal{I}_h(u_h) - (g_1^T + \alpha e_n^T H^{-1} + \beta e_0^T H^{-1}) H(u_h - u) \\
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r_w &:= D u_h - w_h + H^{-1} e_0 (e_0^T u_h - \mathcal{U}_L)
\end{aligned}$$

Noting that $\mathcal{U}_L = e_0^T u$ and $\mathcal{W} = e_n^T w$ and that the residuals are equal to zero, we obtain:

$$r_u := f + D(\Gamma w) - H^{-1}(e_0 e_0^T H^{-1}) H(u_h - u) - H^{-1}(D^T \Gamma^T + e_0 e_0^T \gamma_0 H^{-1}) H(w_h - w)$$

$$r_w := Du - w - H^{-1}(D^T - e_n e_n^T H^{-1}) H(u_h - u) - (w_h - w)$$

Linear elliptic PDEs

Proof (cont.): Let us take now $v_h, z_h \in \mathbb{R}^{n+1}$ vectors that will behave as Lagrange multipliers associated to r_u and r_w. If we take the SBP inner product with r_u and r_w and subtract the result from the functional, reordering it we obtain:

$$\begin{aligned}\mathcal{I}(\mathcal{U}) &= \mathcal{I}_h(u_h) - (g_1^T + \alpha e_n^T H^{-1} + \beta e_0^T H^{-1}) H(u_h - u) \\ &\quad - (g_2^T + \beta\gamma_0 e_0^T H^{-1}) H(w_h - w) - v_h^T H r_u - z_h^T H r_w + \mathcal{O}(h^{2s})\end{aligned}$$

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&\quad + \left[z_h^T D^T - g_1^T - (z_h^T e_n + \alpha) e_n^T H^{-1} + (v_h^T e_0 - \beta) e_0^T H^{-1} \right] H(u_h - u) \\
&\quad + \left[v_h^T D^T \Gamma^T - g_2^T + z_h^T + (v_h^T e_0 - \beta) e_0^T \gamma_0 H^{-1} \right] H(w_h - w) + \mathcal{O}(h^{2s})
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\end{aligned}$$

Now we choose v_h and z_h as the discrete adjoint variables, satisfying:

$$\begin{aligned}
D z_h &= g_1 + H^{-1} e_n (e_n^T z_h + \alpha) - H^{-1} e_0 (e_0^T v_h - \beta) \\
-z_h &= \Gamma D v_h - g_2 + H^{-1} \gamma_0 e_0 (e_0^T v_h - \beta)
\end{aligned}$$

Linear elliptic PDEs

Proof (end):
That is precisely the SBP-SAT discretization of the corresponding continuous adjoint problem:

$$\begin{cases} \dfrac{\partial \mathcal{Z}}{\partial x} = \mathcal{G}_1, \ x \in \Omega \\ -\mathcal{Z} = \gamma \dfrac{\partial \mathcal{V}}{\partial x} - \mathcal{G}_2, \ x \in \Omega \\ \mathcal{V}(0) = \beta \\ \mathcal{Z}(1) = -\alpha \end{cases}$$

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Knowing that the truncation error of the SBP-SAT discretization for v_h is $(s+1)$-accurate and for z_h, at least, s-accurate:

M. Svärd and J. Nordström, *On the order of accuracy for difference approximations of initial boundary value problems*, Journal of Computational Physics, vol. 218, 2006, pp. 333-352.

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■

Outline

Curvilinear grids

SBP operators can be used in higher dimension applying one-dimensional operators to each spatial direction independently. For example, in the $[0,1]^3$ cubic domain with a uniform discretization ($x_{ijk} = \frac{1}{h}(i,j,k),\ \ i,j,k = 0,...,n$), the SBP operator for $\partial/\partial x$ is given by:

$$D_x = I \otimes I \otimes D$$

where D is the one-dimensional SBP operator.

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What happens in general?

Curvilinear grids

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Let us take a compact set $\Omega_x \subset \mathbb{R}^2$ such that there exists an invertible transformation

$$\mathcal{T}(x,y) = (\xi(x,y), \eta(x,y))$$

of class $\mathcal{C}^{2s}$ that maps Ω_x to $\Omega_\xi = [0,1]^2$.

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$$\iint_{\Omega_x} \mathcal{U} dxdy = \iint_{\Omega_\xi} \mathcal{U}\mathcal{J} d\xi d\eta$$

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$$\iint_{\Omega_x} \mathcal{U} dxdy = \iint_{\Omega_\xi} \mathcal{U}\mathcal{J} d\xi d\eta$$

where

$$\mathcal{J} := \left| \frac{\partial(x,y)}{\partial(\xi\eta)} \right| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}$$

Curvilinear grids

Now, we approximate the Jacobian by:

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Assuming unknowns are ordered first in the ξ direction and then in the η one, we can use:

$$D_\xi = (I \otimes D) \quad \text{and} \quad D_\eta = (D \otimes I)$$

Curvilinear grids

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$$D_\xi = (I \otimes D) \quad \text{and} \quad D_\eta = (D \otimes I)$$

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Curvilinear grids

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Theorem

Let Ω_x and Ω_ξ be connected compact subsets of $\mathbb{R}^2$. Furthermore, let $\mathcal{T} : \Omega_x \longrightarrow \Omega_\xi$ be a $\mathcal{C}^{2s}$ invertible mapping. For grids based on the uniform discretization of Ω_ξ and mapped to Ω_x using $\mathcal{T}^{-1}$, for $\mathcal{U} \in \mathcal{C}^{2s}$ the quadrature $\mathcal{I}_h(u) = u(H \otimes H)J$ is a $2s$-order accurate estimate of the integral $\int\int_{\Omega_x} \mathcal{U} dxdy$.

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Transforming this to Ω_ξ, we obtain:

$$\frac{\partial}{\partial \xi}(\lambda_\xi \mathcal{U}) + \frac{\partial}{\partial \eta}(\lambda_\eta \mathcal{U}) = \mathcal{F}\mathcal{J}, \ \ \forall (\xi, \eta) \in \Omega_\xi$$

where

$$\lambda_\xi = \frac{\partial y}{\partial \eta}\lambda_x - \frac{\partial x}{\partial \eta}\lambda_y \quad \text{and} \quad \lambda_\eta = -\frac{\partial y}{\partial \xi}\lambda_x + \frac{\partial x}{\partial \xi}\lambda_y$$

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We approximate the Jacobian as before and velocity by:

$$\Lambda_\xi = \operatorname{diag}(\operatorname{diag}(\lambda_x(x_{ij}, y_{ij}))D_\eta y - \operatorname{diag}(\lambda_y(x_{ij}, y_{ij}))D_\eta x)$$

$$\Lambda_\eta = \operatorname{diag}(-\operatorname{diag}(\lambda_x(x_{ij}, y_{ij}))D_\xi y + \operatorname{diag}(\lambda_y(x_{ij}, y_{ij}))D_\xi x)$$

Curvilinear grids

The SBP-SAT discretizations of the primal and adjoint equations are:

$$\begin{aligned} D_\xi(\Lambda_\xi u_h) + D_\eta(\Lambda_\eta u_h) = (J \otimes f) &- \left[I \otimes (H^{-1} e_0 e_0^T)\right] \Lambda_\xi (u_h - u) \\ &- \left[(H^{-1} e_0 e_0^T) \otimes I\right] \Lambda_\eta (u_h - u) \end{aligned}$$

$$\begin{aligned} -\Lambda_\xi D_\xi v_h - \Lambda_\eta D_\eta v_h = (J \otimes g) &- \Lambda_\xi \left[I \otimes (H^{-1} e_n e_n^T)\right] (v_h - v) \\ &- \Lambda_\eta \left[(H^{-1} e_n e_n^T) \otimes I\right] (v_h - v) \end{aligned}$$

Repeating the one-dimensional reasoning, it can be shown that both have truncation error of $\mathcal{O}(h^{2s})$ in the interior and $\mathcal{O}(h^s)$ on the boundary, as required for superconvergence of functional estimates.

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- Help reduce communication in parallel solution algorithms
- Lead to a linearly time-stable scheme

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Thanks for your attention!