Asymptotically optimal scheduling in a parallel server

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Based on joint work with
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Introduction

Parallel-server model

- Computer systems (cycle stealing)
- Call centers (specialization)

\[
\begin{align*}
\text{Class-1 users} & \quad \text{Class-2 users} \\
\text{Parallel-server model} & \quad \text{Parallel-server model}
\end{align*}
\]

\[
\begin{align*}
c_1 & \quad 1-c_2 \\
1-c_1 & \quad c_2
\end{align*}
\]

\[
c_i < 1, \quad i = 1, 2 \\
c_1 + c_2 > 1
\]

\[
\begin{align*}
1 & \quad (c_1, c_2) \\
S & \quad 1 \quad \text{Capacity class 1}
\end{align*}
\]
Introduction

Parallel resource assignment

Wireless networks (interference)
➢ Data transmissions in downlink

Base station 1

Class-1 users

Base station 2

Class-2 users

\[
(c_1, c_2) \quad \text{Capacity class 2}
\]

\[
S
\]

\[
c_i < 1, \quad i = 1, 2
\]

\[
c_1 + c_2 > 1
\]
Model description

• 2 classes of users
• Poisson arrivals with rate $\lambda_i$
• Exponentially distributed service requirements with mean $1/\mu_i$
• Traffic load of class $i$: $\rho_i = \frac{\lambda_i}{\mu_i}$
• $N_i(t)$: number of class-$i$ users at time $t$

Policy $\pi$

Capacity $s_i(\pi)(t)$ given to class $i$ at time $t$
under policy $\pi$, with $s(\pi)(t) \in S$
Markovian formulation

Markov process \( \vec{N}^\pi(t) = (N_1^\pi(t), N_2^\pi(t)) \) with transition rates:

\[
\begin{align*}
(N_1, N_2) &\rightarrow (N_1, N_2) + e_j \quad \text{with rate } \lambda_j \\
(N_1, N_2) &\rightarrow (N_1, N_2) - e_j \quad \text{with rate } \mu_j s_j^\pi(t) \\
\text{and } (s_1^\pi(t), s_2^\pi(t)) &\in S
\end{align*}
\]

The stochastic process \( \vec{N}^\pi(t) \) depends on the chosen policy \( \pi \)!
Literature on parallel-server model

Analysis under a specific policy

- Fayolle & Iasnogorodski ’79 (coupled processor)
- Cohen & Boxma ’83 (coupled processor)
- Squillante, et al ’01, ‘06
- Harchol-Balter, et al ‘05

On optimal scheduling no results exist: except for a limiting heavy-traffic regime

- Harrison, Lopez ’99
- Ata, Kumar ‘05
- Bell, Williams ‘01-’05 (Threshold-based policies are optimal)
- Mandelbaum, Stolyar ’04 (Max-Weight policies are optimal)
Goal

Find a policy that minimizes “in some sense” $N_1(t) + N_2(t)$

- Minimizing $E(N_1 + N_2)$ is equivalent to minimizing mean waiting time
- Policy can only base action on the state of the system $(N_1(t), N_2(t))$

Outline of the talk

1. (Approximately) optimal policies
   - Exact analysis
   - Asymptotic analysis (fluid-scaling techniques)

2. Numerical evaluation
   - Compare with heavy-traffic optimal policies

3. Future research
Stability

Definition: policy $\pi$ gives a **stable system** if $\overrightarrow{N}(t)$ is positive recurrent.

- Stability conditions strongly depend on the employed policy

Examples:
- Always serve class 1 individually. If class 1 is not there, serve class 2.
- Serve classes 1 and 2 in parallel, whenever possible (coupled processors).

Research mainly focused on finding policies that give maximum stability.

Optimal scheduling
- **Use resources efficiently** in order to avoid an unstable system!
- Minimize number of users
Possible trade-off

Markov process $\overrightarrow{N}^\pi(t) = (N_1^\pi(t), N_2^\pi(t))$ with transition rates:

$\begin{align*}
(N_1, N_2) &\rightarrow (N_1, N_2) + e_j \text{ with rate } \lambda_j \\
(N_1, N_2) &\rightarrow (N_1, N_2) - e_j \text{ with rate } \mu_j s_j^\pi(t) \\
\text{and } (s_1^\pi(t), s_2^\pi(t)) &\in S
\end{align*}$

Maximize:

- Instantaneous departure rate of users
  $$\mu_1 s_1^\pi(t) + \mu_2 s_2^\pi(t)$$
- The total capacity used of the system
  $$s_1^\pi(t) + s_2^\pi(t)$$
Optimality result

Assume \( \max(\mu_1, \mu_2) \leq \mu_1 c_1 + \mu_2 c_2 \).

Serving classes 1 and 2 in parallel:

- maximizes the departure rate of users, since \( \mu_1 c_1 + \mu_2 c_2 \geq \mu_i \)
- maximizes the total capacity used, since \( c_1 + c_2 > 1 \)

**Proposition:** Assume \( \max(\mu_1, \mu_2) \leq \mu_1 c_1 + \mu_2 c_2 \). The policy \( \tilde{\pi} \) that always serves classes 1 and 2 in parallel is stochastically optimal, i.e.,

\[
N_{1}^{\tilde{\pi}}(t) + N_{2}^{\tilde{\pi}}(t) \leq_{st} N_{1}^{\pi}(t) + N_{2}^{\pi}(t),
\]

\[
P(N_{1}^{\tilde{\pi}}(t) + N_{2}^{\tilde{\pi}}(t) > s) \leq P(N_{1}^{\pi}(t) + N_{2}^{\pi}(t) > s) \text{ for all } s \geq 0, t \geq 0
\]

**Proof** by dynamic programming
Dynamic programming

Assume w.l.o.g. $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + c_1 \mu_1 + c_2 \mu_2 = 1$.

Define

$$V_{k+1}(x) = \min_\pi \mathbb{E}(C(N^\pi(k + 1)) | N(0) = x)$$

$$= \min_\pi \mathbb{P}(N_1^\pi(k + 1) + N_2^\pi(k + 1) > y | N(0) = x)$$

After uniformization, the **dynamic programming equation** is given by:

$$V_0(x) = C(x)$$

$$V_{k+1}(x) = \lambda_1 V_k(x + e_1) + \lambda_2 V_k(x + e_2)$$

$$+ \min_{s \in S} \left\{ \sum_{i=1,2} 1(x_i > 0) \mu_i s_i V_k(x - e_i) + (1 - \lambda_1 - \lambda_2 - \sum_{i=1,2} 1(x_i > 0) \mu_i s_i) V_k(x) \right\}$$
Dynamic programming

The Dynamic Programming equation allows in general to:

• In some settings, characterize analytically optimal policy

• Find numerically an optimal policy

• Determine the characterization of an optimal policy
Trade-off

Assume $\mu_2 \leq \mu_1 c_1 + \mu_2 c_2 < \mu_1$

- **Serve class 1 individually:**
  Maximizes the **departure rate** of users from the system
  or
- **Serve classes 1 and 2 in parallel**
  Uses **maximum capacity**, since $c_1 + c_2 > 1$

**Trade-off** between these two effects

Goal: minimize $E(N_1 + N_2)$
Characterization of optimal policy

Case: \( \mu_2 \leq \mu_1 c_1 + \mu_2 c_2 \leq \mu_1 \)

Trade-off \( \rightarrow \) As the number of users varies, the system should dynamically switch between different actions

**Proposition:**
A policy that minimizes

\[ E(N_1 + N_2) \]

is characterized by a **switching curve** \( h(N_1) \)

Proof by dynamic programming

Expression of switching curve is impossible to characterize analytically!
Fluid control model

A fluid process is a non-negative solution \( n^\pi(t) = (n_1^\pi(t), n_2^\pi(t)) \) of

\[
\frac{dn_i^\pi(t)}{dt} = \lambda_i - u_i^\pi(t)\mu_i - u_c^\pi(t)\mu_i c_i, \quad i = 1, 2,
\]

at regular points

with

\[
u_1^\pi(v) + u_2^\pi(v) + u_c^\pi(v) \leq 1,
\]

\[
u_j^\pi(v) \geq 0, \quad j = 1, 2, c,
\]

Minimize

\[
\int_0^\infty (n_1^\pi(t) + n_2^\pi(t))dt
\]

 Fluid model only takes into account the mean drifts of the stochastic model
Minimize
\[ \int_0^\infty (n_1^\pi(t) + n_2^\pi(t)) dt \]
with
\[ \frac{dn_i^\pi(t)}{dt} = \lambda_i - u_i^\pi(t)\mu_i - u_c^\pi(t)\mu_ic_i, \quad i = 1, 2, \]
\[ u_1^\pi(v) + u_2^\pi(v) + u_c^\pi(v) \leq 1, \]
\[ u_j^\pi(v) \geq 0, \quad j = 1, 2, c, \quad n_i^\pi(t) \geq 0, \quad i = 1, 2. \]

**Lemma:** An optimal control \( u^*(t) \) exists (with \( n^*(t) \) the corresponding trajectory) and for all \( D \geq H(n_1 + n_2) \)
\[ \min \int_0^D (n_1(t) + n_2(t)) dt = \int_0^D (n_1^*(t) + n_2^*(t)) dt = \int_0^\infty (n_1^*(t) + n_2^*(t)) dt, \]
Optimal fluid control

Result:

If $\mu_2 \leq \mu_1 c_1 + \mu_2 c_2 < \mu_1$, then the **optimal fluid control** is characterized by the switching curve $h(n_1)$:

\[
    h(n_1) = \alpha \frac{\mu_2}{\mu_1} n_1, \quad \text{if } \rho_1 \leq c_1
\]

and

\[
    h(n_1) = \infty, \quad \text{if } \rho_1 \geq c_1
\]

\[
\alpha := \max\left(0, \frac{c_2 - \rho_2}{c_1 - \rho_1} + \frac{c_1}{c_1 + c_2 - 1} \times \frac{1 - \rho_2 - \frac{\rho_1}{c_1}(1 - c_2)}{c_1 - \rho_1} \times \frac{\mu_1 - c_1 \mu_1 - c_2 \mu_2}{\mu_2}\right)
\]
Optimality result

\[ \frac{dn_i(t)}{dt} = \lambda_i - u_i(t)\mu_i - u_c(t)\mu_i c_i, \; i = 1, 2, \]

Optimal trajectory \( n^*(t) \) when \( \rho_1 < c_1 \) and \( \rho_2 > c_2 \)
Fluid control ↔ stochastic model

Goal: translate optimal fluid control to an approximately optimal policy for the stochastic model

Consider fluid scaling of the stochastic process

\[ N_i^r(0) = r n_i, \quad i = 1, 2, \text{ with } r \in \mathbb{N}. \]

\[ N_i^{\pi,r}(t) := \frac{N_i^{\pi}(rt)}{r} \quad \text{and} \quad T_j^{\pi,r}(t) := \frac{T_j^{\pi}(rt)}{r}. \]
Fluid control ↔ stochastic model

**Goal:** translate optimal fluid control to an approximately optimal policy for the stochastic model

Consider **fluid scaling** of the stochastic process

\[ N_i^r(0) = r n_i, \ i = 1, 2, \text{ with } r \in \mathbb{N}. \]

\[ N_i^{\pi,r}(t) := \frac{N_i^{\pi,r}(rt)}{r} \quad \text{and} \quad T_j^{\pi,r}(t) := \frac{T_j^{\pi,r}(rt)}{r}. \]

**Lemma**

For almost all sample paths and any sequence \( r_k \), there exists a subsequence \( r_{kl} \) such that

\[
\lim_{l \to \infty} N_i^{\pi,r_{kl}}(t) = N_i^\pi(t), \quad i = 1, 2, \text{ u.o.c.,}
\]

\[
\lim_{l \to \infty} T_j^{\pi,r_{kl}}(t) = T_j^\pi(t), \quad j = 0, 1, 2, c, \text{ u.o.c.,}
\]

with

\[
\frac{dN_i^\pi(t)}{dt} = \lambda_i - \mu_i \frac{dT_i^\pi(t)}{dt} - \mu_i c_i \frac{dT_c^\pi(t)}{dt}
\]

and

\[
\frac{dT_1^\pi(t)}{dt} + \frac{dT_2^\pi(t)}{dt} + \frac{dT_c^\pi(t)}{dt} \leq 1
\]

Same equation as fluid control model!
**Proposition:** Consider the stochastic model and assume $\rho_1 < c_1$.

For the policy $\pi^*$ with switching curve $h(n_1) = \alpha \frac{\mu_2}{\mu_1} n_1$ we have

\[ \frac{dT_{1}^{\pi^*}(t)}{dt} = 1, \text{ if } N_{2}^{\pi^*}(t) < \alpha \frac{\mu_2}{\mu_1} N_{1}^{\pi^*}(t), \]
\[ \frac{dT_{c}^{\pi^*}(t)}{dt} = 1, \text{ if } N_{2}^{\pi^*}(t) \geq \alpha \frac{\mu_2}{\mu_1} N_{1}^{\pi^*}(t) \text{ and } N_{1}^{\pi^*}(t) > 0, \]
\[ \frac{dT_{c}^{\pi^*}(t)}{dt} = \frac{\rho_1}{c_1} \text{ and } \frac{dT_{2}^{\pi^*}(t)}{dt} = 1 - \frac{\rho_1}{c_1}, \text{ if } N_{1}^{\pi^*}(t) = 0 \text{ and } N_{2}^{\pi^*}(t) > 0, \]

Since

\[ \frac{dN_{i}^{\pi^*}(t)}{dt} = \lambda_i - \mu_i \frac{dT_{i}^{\pi^*}(t)}{dt} - \mu_i c_i \frac{dT_{c}^{\pi^*}(t)}{dt} \]

we obtain

\[ N_{i}^{\pi^*}(t) = n^*(t) \]
Definition: A policy $\pi^*$ is called asymptotically fluid optimal if

$$\lim_{r \to \infty} \mathbb{E}\left( \int_0^D (N_{1}^{\pi^*,r}(t) + N_{2}^{\pi^*,r}(t))dt \right) = \int_0^\infty (n_1^*(t) + n_2^*(t))dt, \quad \text{with} \quad D \geq H(n_1 + n_2),$$

Let policy $\pi^*$ have the linear switching curve $h(n_1) = \alpha \frac{\mu_2}{\mu_1} n_1$

Using that $\lim_{l \to \infty} N_{i_l}^{\pi^*,\pi_k l}(t) = N_{\pi^*}(t) = n^*(t)$, we have the following result

Result: Policy $\pi^*$ is asymptotically optimal
Numerical example, $\rho_1 < c_1$

Performance in terms of $\mathbb{E}(N_1 + N_2)$

- Optimal policy: obtained numerically by value iteration and truncating state space
- Good match of the asymptotically optimal switching curve with the optimal policy

$$h(n_1) = d \cdot n_1$$
Case $\rho_1 > c_1$

Assume $\rho_1 > c_1$ and $\mu_2 \leq \mu_1 c_1 + \mu_2 c_2 \leq \mu_1$, so $h(n_1) = \infty$.

Optimal policy for fluid model:
Always serve class 1 individually

Stochastic model:
Always serve class 1 individually
$\rightarrow$ unstable if $\rho_1 + \rho_2 > 1$
Case $\rho_1 > c_1$, stochastic model

Result: Policy with exponential switching curve $h(N_1) = e^{N_1/\gamma}$ is asymptotically optimal with $\gamma > 0$ large enough.

Heuristic (based on second order analysis): $\gamma = 1/\ln(\frac{\rho_1}{c_1})$. 
Numerical example, $\rho_1 > c_1$

Performance in terms of $\mathbb{E}(N_1 + N_2)$

- Our rule of thumb: works well!

- Threshold policies can be very good as well. However, no good expressions for close-to-optimal threshold exists.
Numerical example

Max-Weight policies (Stolyar, Mandelbaum)

Max-Weight policy is 15-30% worse than our fluid-based policy
Ongoing work

We found policies that are asymptotically fluid optimal, i.e.,

\[
\lim_{r \to \infty} \mathbb{E}\left( \int_0^D (\overline{N}_1^{\pi_*} r(t) + \overline{N}_2^{\pi_*} r(t)) dt \right) = \int_0^\infty (n_1^*(t) + n_2^*(t)) dt, \quad \text{with} \quad D \geq H(n_1 + n_2),
\]

**Question:** determine the rate of convergence.

Preliminary results: \(O(1/\sqrt{r})\) or \(O(1/r)\).
Ongoing work

We found policies that are asymptotically fluid optimal, i.e.,

$$
\lim_{r \to \infty} \mathbb{E}\left(\int_0^D (N_1^{\pi^*} + N_2^{\pi^*}) dt\right) = \int_0^\infty (n_1^*(t) + n_2^*(t)) dt, \quad \text{with} \quad D \geq H(n_1 + n_2),
$$

Transient cost (cost until time $D$)

Average performance
Numerically we found that these fluid-based policies are also close to optimal in terms of average performance

$$
\mathbb{E}(N_1 + N_2) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left(\int_0^T (N_1(t) + N_2(t)) dt\right)
$$

Question: Can we derive bounds on the average performance?
For example,

$$
\min_{\pi} \mathbb{E}\left(N_1^\pi + N_2^\pi\right) \leq \mathbb{E}\left(N_1^{\pi^*} + N_2^{\pi^*}\right) \leq (1 + g(\rho, \mu)) \min_{\pi} \mathbb{E}\left(N_1^\pi + N_2^\pi\right)
$$

Some fluid-based policy
Asymptotically optimal scheduling in a parallel server

Thank you!

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