Finite time horizon versus stationary control

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In various fields of Science, Engineering and Industry control and design issues play often a key role. And often times problems are formulated in long time intervals.

This is for instance the case in celestial mechanics, climate, etc.

And when that occurs several issues become of a key importance:

- Convergence of dynamic controls versus steady-state ones;
- Long-time numerical simulations.
1 Motivation

2 Evolution versus steady state control

3 Long time numerical simulations

4 General conclusions
Motivation

- Engineering and physical processes do not obey a single modeling paradigm. Often times, both time-dependent and steady models are available and appropriate.
- This yields a number of different possibilities when facing optimal design, control and inverse problems.
- What model do we adopt? The time-dependent or the steady state one.
- Steady-state models are often understood as a simplification of the time evolution one, assuming (some times rigorously but most often without proof) that the time-dependent solution stabilizes around the steady-state as $t \to \infty$.

Main question

Does the solution of the time-evolution control (or design or inverse problem) converge as $t \to \infty$ to the control of the steady state problem as well?
This issue is particularly important in aeronautical optimal design,\(^1\) a mature but still rapidly evolving field where huge challenges arise and, in which, in particular, many problems related to design and control are still widely open.

Often times people employ steady state models and the corresponding control ones while, from a mathematical point of view, the evolution problem is better understood.

The reason for this is very simple: In the context of Nonlinear PDE steady state problems are hard to solve. In particular uniqueness is hard to prove. Accordingly sensitivity analysis is difficult as well. By the contrary, for evolution problems, under suitable assumptions on the nonlinearity, initial-boundary value problems are uniquely solvable, solutions depending smoothly on the data.

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\(^1\)A. Jameson. "Optimization Methods in Computational Fluid Dynamics (with Ou, K.), Encyclopedia of Aerospace Engineering, Edited by Richard Blockley and Wei Shyy, John Wiley Sons, Ltd., 2010."
Although the idea can be traced back to John von Neumann in 1945, Lionel W. McKenzie traces the term to Robert Dorfman, Paul Samuelson, and Robert Solow’s ”Linear Programming and Economics Analysis” in 1958, referring to an American English word for a Highway:

... It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.
Mainly motivated by applications to economic models and game theory there is a literature concerned with this kind of stationary behavior in the transient time for long horizon control problems. In that context, such type of result goes under the name of *turnpike property* which was mostly investigated in the finite dimensional case.

Consider the finite dimensional dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0 \end{cases} \quad (1)$$

where $A \in M(N, N)$, $B \in M(N, M)$, the control $u \in L^2(0, T; \mathbb{R}^M)$, and $x_0 \in \mathbb{R}^N$.

Given a matrix $C \in M(N, N)$, and some $x^* \in \mathbb{R}^N$, consider the optimal control problem

$$\min_u J^T(u) = \frac{1}{2} \int_0^T (|u(t)|^2 + |C(x(t) - x^*)|^2) dt.$$ 

There exists a unique optimal control $u(t)$ in $L^2(0, T; \mathbb{R}^M)$, characterized by the optimality condition

$$u = -B^*p, \quad \begin{cases} -p_t + A^*p = C^*C(x - x^*) \\ p(T) = 0 \end{cases} \quad (2)$$
Similar problems can be formulated in the context of PDE (heat equations, elasticity, fluids,...). In that case the equation evolves in $\Omega$ while the control is localized in $\omega \subset \Omega$. 
The same problem can be formulated for the steady-state model

\[ Ax = Bu. \]

Then there exists a unique minimum \( \bar{u} \), and a unique optimal state \( \bar{x} \), of the stationary "control problem"

\[
\min_u J_s(u) = \frac{1}{2}(|u|^2 + |C(x - x^*)|^2), \quad Ax = Bu, \tag{3}
\]

which is nothing but a constrained minimization in \( \mathbb{R}^N \); and by elementary calculus, the optimal control \( \bar{u} \) and state \( \bar{x} \) satisfy

\[ A\bar{x} = B\bar{u}, \quad \bar{u} = -B^\ast \bar{p}, \quad \text{and} \quad A^\ast \bar{p} = C^\ast C(\bar{x} - x^*). \]
We assume that

\[ \text{The pair } (A, B) \text{ is controllable,} \quad (4) \]

or, equivalently, that the matrices \( A, B \) satisfy the Kalman rank condition

\[ \text{Rank} \begin{bmatrix} B & AB & A^2B & \ldots & A^{N-1}B \end{bmatrix} = N. \quad (5) \]

Then there exists a linear stabilizing feedback law \( L \in M(M, N) \) and \( c, \mu > 0 \) such that

\[
\begin{cases}
  x_t + Ax = B(Lx) \\
x(0) = x_0
\end{cases} \implies |x(t)| \leq ce^{-\mu t}|x_0| \quad \forall t > 0. \quad (6)
\]

Concerning the cost functional, we assume that the matrix \( C \) is such that

\[ \text{The pair } (A, C) \text{ is observable} \quad (7) \]

which means that the following algebraic condition holds:

\[ \text{Rank} \begin{bmatrix} C & CA & CA^2 & \ldots & CA^{N-1} \end{bmatrix} = N. \quad (8) \]
Under the above controllability and observability assumptions, we have the following result.

**Theorem**

Assume that (5) and (8) hold true. Then we have

\[
\frac{1}{T} \min_{u \in L^2(0,T)} J^T \quad \text{as} \quad T \to \infty \quad \min_{u \in \mathbb{R}^N} J_s
\]

and

\[
\frac{1}{T} \int_0^T (|u(t) - \tilde{u}|^2 + |C(x(t) - \tilde{x})|^2) \, dt \to 0
\]

where \( \tilde{u} \) is the optimal control of \( J_s \) and \( \tilde{x} \) the corresponding optimal state.

In particular, we have

\[
\frac{1}{(b-a)T} \int_{aT}^{bT} x(t) \, dt \to \tilde{x} \quad , \quad \frac{1}{(b-a)T} \int_{aT}^{bT} u(t) \, dt \to \tilde{x}
\]

for every \( a, b \in [0,1] \).
Proof.

We use the optimality conditions defining the adjoint states $p$ and $\bar{p}$, which give

\[
\begin{cases}
(x - \bar{x})_t + A(x - \bar{x}) = B(u - \bar{u}) \\
u - \bar{u} = -B^*(p - \bar{p}) \\
-(p - \bar{p})_t + A^*(p - \bar{p}) = C^*C(x - \bar{x})
\end{cases}
\]

From the optimality system we get

\[
[(x - \bar{x})(p - \bar{p})]_t = B(u - \bar{u})(p - \bar{p}) - |C(x - \bar{x})|^2
\]

which implies

\[
\int_0^T (|u - \bar{u}|^2 + |C(x - \bar{x})|^2) \, dt = [(x_0 - \bar{x})(p(0) - \bar{p})] + [(x(T) - \bar{x})\bar{p}].
\]
If \((A, B)\) is controllable, then there exists \(c\), independent of \(T\), such that, for every \(f \in L^2(0, T; \mathbb{R}^N)\), \(q_T \in \mathbb{R}^N\), the solution of

\[
\begin{cases}
-q_t + A^* q = f \\
q(T) = q_T
\end{cases}
\]  

satisfies, with \(\mu\) is as above,

\[
|q(0)|^2 \leq c \left[ \int_0^T |B^* q|^2 dt + \int_0^T |f|^2 dt + e^{-2\mu T} |q_T|^2 \right].
\]  

Indeed, multiplying the adjoint equation by \(x\) of (6), we get

\[
q(0) \cdot x_0 = q_T \cdot x(T) - \int_0^T q(x_t + Ax) \, dt + \int_0^T f \cdot x \, dt
\]

\[
= q_T \cdot x(T) - \int_0^T B^* q \cdot Lx \, dt + \int_0^T f \cdot x \, dt,
\]

and using the exponential decay of \(x(t)\) we obtain

\[
|q(0) \cdot x_0| \leq C|x_0| \left[ \int_0^T |B^* q|^2 dt + \int_0^T |f|^2 dt \right]^{\frac{1}{2}} + C|x_0| e^{-\mu T} |q_T|
\]

which suffices with \(x_0 = q(0)\).
Using the observability inequality (10) we have

\[
|p(0) - \bar{p}| \leq c \left[ \left( \int_0^T |C(x - \bar{x})|^2 \, dt \right)^{\frac{1}{2}} + \left( \int_0^T |B^*(p - \bar{p})|^2 \, dt \right)^{\frac{1}{2}} + |\bar{p}| \right].
\]

(11)

Similarly, in the equation of \( x - \bar{x} \) we use the observability inequality for \((A, C)\) which is ensured by (8):

\[
|x(T) - \bar{x}| \leq c \left( \int_0^T |u - \bar{u}|^2 \, dt + \int_0^T |C(x(t) - \bar{x})|^2 \, dt + |x_0 - \bar{x}|^2 \right)^{\frac{1}{2}}.
\]

(12)
Hence

\[
\int_0^T \left( |u - \bar{u}|^2 + |C(x - \bar{x})|^2 \right) dt \leq c
\]  

(13)

by the previous estimates (11) and (12).

We conclude

\[
\frac{1}{T} \int_0^T \left( |u - \bar{u}|^2 + |C(x - \bar{x})|^2 \right) dt \leq \frac{C}{T} \rightarrow 0.
\]

This of course also implies the convergence of the averaged minimum level to the stationary minimum.
Following [CLLP]², if $B^*$ and $C$ are coercive we also have

$$\|u - \bar{u}\|^2 + |C(x - \bar{x})|^2 = |B^*(p - \bar{p})|^2 + |C(x - \bar{x})|^2 \geq \gamma \left( |p - \bar{p}|^2 + |x - \bar{x}|^2 \right)$$

hence we deduce from the optimality system

$$[ (x - \bar{x})(p - \bar{p}) ]_t = - |B^*(p - \bar{p})|^2 - |C(x - \bar{x})|^2 \leq - \gamma |(x - \bar{x})(p - \bar{p})|,$$

for some $\gamma > 0$. Since $(x - \bar{x})(p - \bar{p})$ is bounded at $t = 0$ and $t = T$ due to (11), (12) and (13), we obtain

$$- e^{-\gamma(T-t)} K \leq [(x - \bar{x})(p - \bar{p})](t) \leq Ke^{-\gamma t}$$

for some $K > 0$. Integrating we get

$$\int_{aT}^{bT} \left( \|u - \bar{u}\|^2 + |x - \bar{x}|^2 \right) ds \leq K \left( e^{-\gamma aT} + e^{-\gamma (1-b)T} \right)$$

which implies an exponential rate of convergence.

The same results hold for linear PDE and in particular heat and wave equations.

Note that the problem in the time interval $[0, T]$ as $T \to \infty$ can be rescaled into the fixed time interval $[0, 1]$ by the change of variables $t = Ts$:

$$\varepsilon x_s + Ax = Bu, \quad s \in [0, 1].$$

In the limit as $\varepsilon \to 0$ the steady-state equation emerges:

$$Ax = Bu.$$

This becomes a classical singular perturbation control problem. Note however that, in this setting, the role that the controllability and observability properties of the system play is much less clear than when dealing with it as $T \to \infty$.

The problem in widely open for nonlinear models.
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Numerical integration of the pendulum

- Euler explicit, \( t=0 \)
- Euler implicit, \( t=0 \)
- Euler symplectic, \( t=0 \)
- Fourth-order RK, \( t=0 \)
Climate modeling is a grand challenge computational problem, a research topic at the frontier of computational science.

Simplified models for geophysical flows have been developed aim to: capture the important geophysical structures, while keeping the computational cost at a minimum.

Although successful in numerical weather prediction, these models have a prohibitively high computational cost in climate modeling.

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Sonic boom

- Goal: the development of supersonic aircraft that are sufficiently quiet so that they can be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when it reaches the ground, (a) it can barely be perceived by the human ear, and (b) it results in disturbances to man-made structures that do not exceed the threshold of annoyance for a significant percentage of the population.

Consider the 1-D conservation law with or without viscosity:

$$u_t + [u^2]_x = \varepsilon u_{xx}, x \in \mathbb{R}, t > 0.$$ 

Then:

- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

$u_M$ is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass $M$ of $u_0$), while $N$ is the so-called hyperbolic N-wave.

In both cases:

$$u(x, t) \sim t^{-1/2} F(x/\sqrt{t}), \ t \to \infty.$$
The rest of this paper is divided as follows: in Section \(v\) we present some classical facts about the numerical approximation of one-dimensional conservation laws and obtain preliminary results that will be used in the proof of the main results of this paper. In Section \(w\) we prove the main result, Theorem \(u\), and we illustrate it in Section \(x\) with a numerical simulation. In Section \(y\) we discuss the approximation through similarity variables and compare the results to the approximations obtained directly from the physical ones. Finally, in Section \(z\) we give some ideas about how to generalize the results to other numerical schemes and to more general fluxes uniformly convex or odd ones.

Preliminaries

In this part, following [\(w\)] and [7], we recall a few of the well-known results about numerical schemes for scalar conservation laws. We obtain some new results that will be used in Section \(w\) in the proof of Theorem \(u\). We restrict our attention to the Burgers equation, where the nonlinear term \(f\) is given by

\[ f(u) = u^2. \]

More general results will be discussed in Section \(y\) for uniformly convex fluxes and odd fluxes. First, given a time step \(\Delta t\) and a uniform spatial grid \(\Delta x\) with space increment \(\Delta x\), we approximate the conservation law

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x\in \mathbb{R}, t>0, \]

by an explicit difference scheme of the form:

\[ u^n_{j+1} = H(u^n_j - k,...,u^n_j + k,\ldots), \quad \forall n \geq 1, j \in \mathbb{Z}, \]

where \(H: \mathbb{R}^{2k+1} \to \mathbb{R}\) is a continuous function and \(u^n_j\) denotes the approximation of the exact solution at the node \(n\Delta t, j\Delta x\). Assuming that there exists a continuous function \(g: \mathbb{R}^{2k} \to \mathbb{R}\) called numerical flux, such that

\[ H(u^n_j - k,...,u^n_j + k) = u^n_0 - \lambda [g(u^n_j - k+1,...,u^n_j - k)] - [g(u^n_j - k,...,u^n_j - k-1)]\]

\[ \lambda = \Delta t/\Delta x, \]

we can proceed to the numerical simulation.
Let us consider now numerical approximation schemes

\[
\begin{align*}
\begin{cases}
  u_{j}^{n+1} = u_{j}^{n} - \frac{\Delta t}{\Delta x} \left( g_{j+1/2}^{n} - g_{j-1/2}^{n} \right), & j \in \mathbb{Z}, n > 0. \\
  u_{j}^{0} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_{0}(x) \, dx, & j \in \mathbb{Z},
\end{cases}
\end{align*}
\]

The approximated solution \( u_{\Delta} \) is given by

\[
u_{\Delta}(t, x) = u_{j}^{n}, \quad x_{j-1/2} < x < x_{j+1/2}, \quad t_{n} \leq t < t_{n+1},
\]

where \( t_{n} = n\Delta t \) and \( x_{j+1/2} = (j + \frac{1}{2})\Delta x \).

Is the large time dynamics of these discrete systems, a discrete version of the continuous one?
3-point conservative schemes

1. Lax-Friedrichs

\[ g^{LF}(u, v) = \frac{u^2 + v^2}{4} - \frac{\Delta x}{\Delta t} \left( \frac{v - u}{2} \right), \]

2. Engquist-Osher

\[ g^{EO}(u, v) = \frac{u(u + |u|)}{4} + \frac{v(v - |v|)}{4}, \]

3. Godunov

\[ g^G(u, v) = \begin{cases} \min_{w \in [u,v]} \frac{w^2}{2}, & \text{if } u \leq v, \\ \max_{w \in [v,u]} \frac{w^2}{2}, & \text{if } v \leq u. \end{cases} \]
We can rewrite three-point monotone schemes in the form

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{4\Delta x} = R(u_j^n, u_{j+1}^n) - R(u_{j-1}^n, u_j^n)
\]

where the numerical viscosity \( R \) can be defined in a unique manner as

\[
R(u, v) = \frac{Q(u, v)(v - u)}{2} = \frac{\lambda}{2} \left( \frac{u^2}{2} + \frac{v^2}{2} - 2g(u, v) \right).
\]

For instance:

\[
R^{LF}(u, v) = \frac{v - u}{2},
\]

\[
R^{EO}(u, v) = \frac{\lambda}{4} (v|v| - u|u|),
\]

\[
R^{G}(u, v) = \begin{cases} 
\frac{\lambda}{4} \text{sign}(|u| - |v|)(v^2 - u^2), & v \leq 0 \leq u, \\
\frac{\lambda}{4} (v|v| - u|u|), & \text{elsewhere}.
\end{cases}
\]
These three schemes are well-known to satisfy the following properties:

- They converge to the entropy solution
- They are monotonic
- They preserve the total mass of solutions
- They are OSLC consistent:
  \[ \frac{u^n_{j-1} - u^n_{j+1}}{2\Delta x} \leq \frac{2}{n\Delta t} \]
- \(L^1 \to L^\infty\) decay with a rate \(O(t^{-1/2})\)
- Similarly they verify uniform \(BV_{loc}\) estimates

But do they capture correctly the asymptotic behavior of solutions as \(t \to \infty\)?
Main result: Viscous effective behavior

Theorem (Lax-Friedrichs scheme)

Consider $u_0 \in L^1(\mathbb{R})$ and $\Delta x$ and $\Delta t$ such that $\lambda \left| u^n \right|_{\infty, \Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solution $u_\Delta$ given by the Lax-Friedrichs scheme satisfies

$$\lim_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p})} \left| u_\Delta(t) - w(t) \right|_{L^p(\mathbb{R})} = 0,$$

where the profile $w = w_{M_\Delta}$ is the unique solution of

$$\begin{cases} w_t + \left( \frac{w^2}{2} \right)_x = \frac{(\Delta x)^2}{2} w_{xx}, & x \in \mathbb{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$. 
Long time numerical simulations

Why?

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{4\Delta x} = R^{LF}(u_j^n, u_{j+1}^n) - R^{LF}(u_{j-1}^n, u_j^n)
\]

with

\[
R^{LF}(u, v) = \frac{v - u}{2}.
\]

Thus

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{4\Delta x} \sim \frac{1}{2} \left[ u_{j+1}^n + u_{j-1}^n - 2u_j^n \right] \sim \frac{(\Delta x)^2}{2} u_{xx}.
\]
Long time numerical simulations

Main result: Inviscid effective behavior

Theorem (Engquist-Osher and Godunov schemes)

Consider $u_0 \in L^1(\mathbb{R})$ and $\Delta x$ and $\Delta t$ such that $\lambda \left| u^n \right|_{\infty, \Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solutions $u_\Delta$ given by Engquist-Osher and Godunov schemes satisfy the same asymptotic behavior but for the hyperbolic $N$-wave $w = w_{p_\Delta, q_\Delta}$ unique solution of

$$
\begin{cases}
  w_t + \left( \frac{w^2}{2} \right)_x = 0, & x \in \mathbb{R}, t > 0, \\
  w(0) = M_\Delta \delta_0, & \lim_{t \to 0} \int_0^x w(t, z) dz = \begin{cases} 
  0, & x < 0, \\
  -p_\Delta, & x = 0, \\
  q_\Delta - p_\Delta, & x > 0,
  \end{cases}
\end{cases}
$$

with $M_\Delta = \int_{\mathbb{R}} u^0_\Delta$ and $p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^x u^0_\Delta(z) dz$ and $q_\Delta = \max_{x \in \mathbb{R}} \int_x^\infty u^0_\Delta(z) dz$. 
Scaling transformation:

\[ u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \]

The asymptotic behavior of \( u(x, t) \) as \( t \to \infty \) is reduced to the analysis of the behavior of the rescaled family \( u_\lambda \) as \( \lambda \to \infty \) but in the finite time horizon \( 0 < t < 1 \).
Example

Let us consider the inviscid Burgers equation with initial data

\[ u_0(x) = \begin{cases} 
-0.05, & x \in [-1, 0], \\
0.15, & x \in [0, 2], \\
0, & \text{elsewhere}.
\end{cases} \]

The parameters that describe the asymptotic N-wave profile are:

\[ M = 0.25, \quad p = 0.05 \quad \text{and} \quad q = 0.3. \]

We take \( \Delta x = 0.1 \) as the mesh size for the interval \([-350, 800]\) and \( \Delta t = 0.5 \). Solution to the Burgers equation at \( t = 10^5 \):
Let us consider the change of variables given by:

\[ s = \ln(t + 1), \quad \xi = x/\sqrt{t + 1}, \quad w(\xi, s) = \sqrt{t + 1} \, u(x, t), \]

which turns the continuous Burgers equation into

\[ w_s + \left( \frac{1}{2} w^2 - \frac{1}{2} \xi w \right)_\xi = 0, \quad \xi \in \mathbb{R}, \ s > 0. \]

The asymptotic profile of the N-wave becomes a steady-state solution:

\[ N_{p,q}(\xi) = \begin{cases} 
\xi, & -\sqrt{2p} < \xi < \sqrt{2q}, \\
0, & \text{elsewhere}, 
\end{cases} \]
Convergence of the numerical solution using Engquist-Osher scheme (circle dots) to the asymptotic N-wave (solid line). We take $\Delta \xi = 0.01$ and $\Delta s = 0.0005$.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$. 
Numerical solution using the Lax-Friedrichs scheme (circle dots), taking $\Delta \xi = 0.01$ and $\Delta s = 0.0005$. The N-wave (solid line) is not reached, as it converges to the diffusion wave.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$. 
Within the class of convergent numerical schemes we have shown the need of discriminating those that are asymptotically correct.

We have shown the significant reduction on the computational cost when using the intrinsic similarity variables.
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Lots to be done on:

- The analysis of how time-evolution control and steady-state one are related for nonlinear problems.
- Similar issues in the context of shape design and inverse problems.
- Development of numerical algorithms preserving large time asymptotics for nonlinear PDEs (other works of our team on dispersive equations, dissipative wave equations,...)
- The two issues overlap when trying to effectively compare numerical approximations.