Switching, sparse and averaged control

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Often control systems are endowed with different actuators. It is then convenient to develop control strategies alternating the use of one and the other so as to minimize the overall cost and fatigue of the system while maintaining the control properties.
What is the best switching strategy to capture or control the dynamics?

- Reduce the cost of control.
- Reduce the fatigue of the system.
- Maximize the observation/control properties of the system.
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Consider the finite dimensional linear control system

\[
\begin{aligned}
&x'(t) = Ax(t) + u_1(t)b_1 + u_2(t)b_2 \\
x(0) = x^0.
\end{aligned}
\]  

(1)

\(x(t) = (x_1(t), \ldots, x_N(t)) \in \mathbb{R}^N\) is the state of the system, \(A\) is a \(N \times N\) matrix, \(u_1 = u_1(t)\) and \(u_2 = u_2(t)\) are two scalar controls and \(b_1, b_2\) are given control vectors in \(\mathbb{R}^N\).
More general and complex systems may also involve switching in the state equation itself:

\[
x'(t) = A(t)x(t) + u_1(t)b_1 + u_2(t)b_2, \quad A(t) \in \{A_1, ..., A_M\}.
\]
Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for control pairs $(u_1, u_2)$ such that the solution of (21) satisfies

$$x(T) = x^1. \quad (2)$$

In the absence of constraints, both controls being allowed to act simultaneously, controllability holds if and only if the Kalman rank condition is satisfied

$$[B, AB, \ldots, A^{N-1}B] = N \quad (3)$$

with $B = (b_1, b_2)$. 
We look for **switching controls:**

\[
    u_1(t)u_2(t) = 0, \quad \text{a.e.} \quad t \in (0, T).
\]  

(4)

*Under the rank condition above, these switching controls always exist.*

This is one of the simplest problems one may consider in a very wide and unexplored, to a large extent, field. For instance, switching and intermittent damping and stabilization:


The classical theory guarantees that the standard controls \((u_1, u_2)\) may be built by minimizing the functional

\[
J(\varphi^0) = \frac{1}{2} \int_0^T \left[ |b_1 \cdot \varphi(t)|^2 + |b_2 \cdot \varphi(t)|^2 \right] dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),
\]

among the solutions of the adjoint system

\[
\begin{cases}
-\varphi'(t) = A^* \varphi(t), & t \in (0, T) \\
\varphi(T) = \varphi^0.
\end{cases}
\]

The rank condition for the pair \((A, B)\) is equivalent to the following unique continuation property for the adjoint system which suffices to show the coercivity of the functional:

\[
b_1 \cdot \varphi(t) = b_2 \cdot \varphi(t) = 0, \quad \forall t \in [0, T] \rightarrow \varphi \equiv 0.
\]
Preassigned switching

Given a partition \( \tau = \{ t_0 = 0 < t_1 < t_2 < \ldots < t_{2N} = T \} \) of the time interval \((0, T)\), consider the functional

\[
J_\tau (\varphi^0) = \frac{1}{2} \sum_{j=0}^{N-1} \int_{t_{2j}}^{t_{2j+1}} |b_1 \cdot \varphi(t)|^2 dt + \frac{1}{2} \sum_{j=0}^{N-1} \int_{t_{2j+1}}^{t_{2j+2}} |b_2 \cdot \varphi(t)|^2 dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0).
\]

Under the same rank condition this functional is coercive too. In fact, in view of the time-analiticality of solutions, the above unique continuation property implies the apparently stronger one:

\[
b_1 \cdot \varphi(t) = 0 \quad t \in (t_{2j}, t_{2j+1}); \quad b_2 \cdot \varphi(t) = 0 \quad t \in (t_{2j+1}, t_{2j+2}) \rightarrow \varphi \equiv 0
\]

and this one suffices to show the coercivity of \( J_\tau \). Thus, \( J_\tau \) has an unique minimizer \( \tilde{\varphi} \) and this yields the controls

\[
u_1(t) = b_1 \cdot \tilde{\varphi}(t), \quad t \in (t_{2j}, t_{2j+1}); \quad u_2(t) = b_2 \cdot \tilde{\varphi}(t), \quad t \in (t_{2j+1}, t_{2j+2})
\]

which are obviously of switching form.
Consider now:

\[
J_s(\varphi^0) = \frac{1}{2} \int_0^T \max \left( |b_1 \cdot \varphi(t)|^2, |b_2 \cdot \varphi(t)|^2 \right) dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0). \tag{6}
\]

**Theorem**

Assume that the pairs \((A, b_2 - b_1)\) and \((A, b_2 + b_1)\) satisfy the rank condition. Then, for all \(T > 0\), \(J_s\) achieves its minimum at least on a minimizer \(\tilde{\varphi}^0\). Furthermore, the switching controllers

\[
\begin{align*}
    u_1(t) &= \tilde{\varphi}(t) \cdot b_1 & \text{when} & \left| \tilde{\varphi}(t) \cdot b_1 \right| > \left| \tilde{\varphi}(t) \cdot b_2 \right| \\
    u_2(t) &= \tilde{\varphi}(t) \cdot b_2 & \text{when} & \left| \tilde{\varphi}(t) \cdot b_2 \right| > \left| \tilde{\varphi}(t) \cdot b_1 \right|
\end{align*}
\]

where \(\tilde{\varphi}\) is the solution of (25) with datum \(\tilde{\varphi}^0\) at time \(t = T\), control the system.

---

The rank condition on the pairs \((A, b_2 \pm b_1)\) is a necessary and sufficient condition for the controllability of the systems

\[x' + Ax = (b_2 \pm b_1) u(t).\] (8)

This implies that the system with controllers \(b_1\) and \(b_2\) is controllable too but the reverse is not true.

The rank conditions on the pairs \((A, b_2 \pm b_1)\) are needed to ensure that the set

\[\{ t \in (0, T) : |\varphi(t) \cdot b_1| = |\varphi(t) \cdot b_2| \}\] (9)

is of null measure, which ensures that the controls in (7) are genuinely of switching form.
Sketch of the proof

There are two key points:

a) Showing that the functional $J_s$ is coercive, i.e.,

$$\lim_{\|\varphi^0\| \to \infty} \frac{J_s(\varphi^0)}{\|\varphi^0\|} = \infty,$$

which guarantees the existence of minimizers.

Coercivity is immediate since

$$|\varphi(t) \cdot b_1|^2 + |\varphi(t) \cdot b_2|^2 \leq 2 \max \left[ |\varphi(t) \cdot b_1|^2, |\varphi(t) \cdot b_2|^2 \right]$$

and, consequently, the functional $J_s$ is bounded below by a functional equivalent to the classical one $J$.

b) Showing that the controls obtained by minimization are of switching form.
The Euler-Lagrange equations associated to the minimization of $J_s$ take the form
\[
\int_{S_1} \tilde{\varphi}(t) \cdot b_1 \psi(t) \cdot b_1 \, dt + \int_{S_2} \tilde{\varphi}(t) \cdot b_2 \psi(t) \cdot b_2 \, dt - x^1 \cdot \psi^0 + x^0 \cdot \psi(0) = 0,
\]
for all $\psi^0 \in \mathbb{R}^N$, where
\[
\begin{align*}
S_1 &= \{ t \in (0, T) : |\tilde{\varphi}(t) \cdot b_1| > |\tilde{\varphi}(t) \cdot b_2| \}, \\
S_2 &= \{ t \in (0, T) : |\tilde{\varphi}(t) \cdot b_1| < |\tilde{\varphi}(t) \cdot b_2| \}.
\end{align*}
\]
(10)

In view of this we conclude that
\[
u_1(t) = \tilde{\varphi}(t) \cdot b_1 \mathbbm{1}_{S_1}(t), \quad u_2(t) = \tilde{\varphi}(t) \cdot b_2 \mathbbm{1}_{S_2}(t),
\]
(11)
where $\mathbbm{1}_{S_1}$ and $\mathbbm{1}_{S_2}$ stand for the characteristic functions of the sets $S_1$ and $S_2$, are such that the switching condition holds and the corresponding solution satisfies the final control requirement.
But for this to be rigorous we need that the set

\[ I = \{ t \in (0, T) : |\tilde{\varphi} \cdot b_1| = |\tilde{\varphi} \cdot b_2| \} \]

is of null measure.

Assume for instance that the set \( I_+ = \{ t \in (0, T) : \tilde{\varphi}(t) \cdot (b_1 - b_2) = 0 \} \) is of positive measure, \( \tilde{\varphi} \) being the minimizer of \( J_s \). The time analyticity of \( \tilde{\varphi} \cdot (b_1 - b_2) \) implies that \( I_+ = (0, T) \). Accordingly \( \tilde{\varphi} \cdot (b_1 - b_2) \equiv 0 \) and, consequently, taking into account that the pair \((A, b_1 - b_2)\) satisfies the Kalman rank condition, this implies that \( \tilde{\varphi} \equiv 0 \). This would imply that

\[ J(\varphi^0) \geq 0, \forall \varphi^0 \in \mathbb{R}^N \]

which may only happen in the trivial situation in which \( x^1 = e^{AT} x^0 \), a trivial situation that we may exclude.
The switching controls we obtain this way are of minimal $L^2(0, T; \mathbb{R}^2)$-norm, the space $\mathbb{R}^2$ being endowed with the $\ell^1$ norm, i.e. with respect to the norm

$$
\|(u_1, u_2)\|_{L^2(0, T; \ell^1)} = \left[ \int_0^T (|\tilde{u}_1| + |\tilde{u}_2|)^2 \, dt \right]^{1/2}.
$$

Switching bang-bang controls are of minimal $L^\infty(0, T; \mathbb{R}^2)$-norm.
Slight variants of these arguments lead to switching controls of different nature, in particular to switching bang-bang controls. For instance, when minimizing the functional

\[ J_{sb}(\varphi^0) = \frac{1}{2} \left[ \int_0^T \max(|\varphi(t) \cdot b_1|, |\varphi(t) \cdot b_2|) \, dt \right]^2 - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0), \]

the controls take the form

\[ u_1(t) = \lambda \, \text{sgn} \left( \tilde{\varphi}(t) \cdot b_1 \right) 1_{S_1}(t); \quad u_2(t) = \lambda \, \text{sgn} \left( \tilde{\varphi}(t) \cdot b_2 \right) 1_{S_2}(t). \]

where

\[ \lambda = \int_0^T \max \left( |\tilde{\varphi}(t) \cdot b_1|, |\tilde{\varphi}(t) \cdot b_2| \right) \, dt. \]
Switching control

Switching control

Classical controls

Bang-bang controls

Switching controls

Switching bang–bang controls

$u_{c_1}$

$u_{c_2}$

$u_{bb_1}$

$u_{bb_2}$

$u_{s_1}$

$u_{s_2}$

$u_{sbb_1}$

$u_{sbb_2}$
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Consider the heat equation in the space interval $\left(0, 1\right)$ with two controls located on the extremes $x = 0, 1$:

\[
\begin{cases}
    y_t - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\
    y(0, t) = u_0(t), \quad y(1, t) = u_1(t), & 0 < t < T \\
    y(x, 0) = y^0(x), & 0 < x < 1.
\end{cases}
\]

We look for controls $u_0, u_1 \in L^2(0, T)$ such that the solution satisfies

\[y(x, T) \equiv 0.\]
To build controls we consider the adjoint system

\[
\begin{cases}
\varphi_t + \varphi_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\
\varphi(0, t) = \varphi(1, t) = 0, & 0 < t < T \\
\varphi(x, T) = \varphi^0(x), & 0 < x < 1.
\end{cases}
\]

It is well known that the null control may be computed by minimizing the quadratic functional

\[
J(\varphi^0) = \frac{1}{2} \int_0^T \left[ |\varphi_x(0, t)|^2 + |\varphi_x(1, t)|^2 \right] dt + \int_0^1 y^0(x)\varphi(x, 0) dx.
\]

The controls obtained this way take the form

\[
u_0(t) = -\hat{\varphi}_x(0, t); \quad u_1(t) = \hat{\varphi}_x(1, t), \quad t \in (0, T)
\]

(12)

where \(\hat{\varphi}\) is the solution associated to the minimizer of \(J\).
For building switching controls we rather consider

\[ J_s(\varphi^0) = \frac{1}{2} \int_0^T \max \left[ |\varphi_x(0, t)|^2, |\varphi_x(1, t)|^2 \right] dt + \int_0^1 y^0(x) \varphi(x, 0) dx. \]

But for this to yield switching controls, the following UC is needed. And it fails because of symmetry considerations!

\[ \text{meas} \{ t \in [0, T] : \varphi_x(0, t) = \pm \varphi_x(1, t) \} = 0. \]
One possible remedy is to consider the functional

$$J_{s,a}(\varphi^0) = \frac{1}{2} \int_0^T \max \left[ |\varphi_x(0, t)|^2, a|\varphi_x(1, t)|^2 \right] dt + \int_0^1 y^0(x)\varphi(x, 0)dx,$$

for some $a \in \mathbb{R}$.

We then need

$$\text{meas} \left\{ t \in [0, T] : \varphi_x(0, t) = \pm a\varphi_x(1, t) \right\} = 0.$$

And this can be assured as soon as $a \neq \pm 1$. 
This strategy yields switching controls for the control problem with two pointwise actuators:

\[
\begin{cases}
  y_t - y_{xx} = u_a(t)\delta_a + u_b(t)\delta_b, & 0 < x < 1, \ 0 < t < T \\
  y(0, t) = y(1, t) = 0, & 0 < t < T \\
  y(x, 0) = y^0(x), & 0 < x < 1,
\end{cases}
\]

under the irrationality condition

\[a \pm b \neq \frac{m}{k}, \ \forall k \geq 1, \ m \in \mathbb{Z}.\]
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\[ \begin{cases} 
  y_t - \Delta y = u_1(x, t)1_{\omega_1} + u_2(x, t)1_{\omega_2} & \text{in } Q \\
  y = 0 & \text{on } \Sigma \\
  y(x, 0) = y^0(x) & \text{in } \Omega.
\end{cases} \]

Switching controls for this problem may be obtained minimizing the functional

\[
J_s(\varphi^0) = \frac{1}{2} \int_0^T \max \left( \int_{\omega_1} |\varphi|^2 dx, \int_{\omega_2} |\varphi|^2 dx \right) dt - \int_0^1 y^0 \varphi(x, 0).
\]

This time the adjoint system is

\[ \begin{cases} 
  \varphi_t + \Delta \varphi = 0 & \text{in } Q \\
  \varphi = 0 & \text{on } \Sigma \\
  \varphi(x, T) = \varphi^0(x) & \text{in } \Omega
\end{cases} \]

and the needed UC property is:

\[
\text{meas}\{ t \in (0, T) : \|\varphi(t)\|_{L^2(\omega_1)} = \|\varphi(t)\|_{L^2(\omega_2)} \} = 0.
\]
This is a non-standard UC problem.

One may expect it to be fulfilled in a “generic” manner.

But a systematic analysis of this issue is to be developed.
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Problem formulation

This time we address the situation where the switching pattern is unknown and can not be regulated. Our goal is then to determine the control at a given \( t \), only based on past informations, in such a way that, at the final time \( T \), the null control property is guaranteed.\(^1\)

Let \( \gamma(\cdot) : \mathbb{R} \to \{0, 1\} \) be a measurable function. Consider the following controlled heat equation:

\[
\begin{cases}
  y_t - \Delta y = \left[ \gamma \chi_{\omega_1} + (1 - \gamma) \chi_{\omega_2} \right] u & \text{in } \Omega \times (0, T), \\
  y = 0 & \text{on } \partial \Omega \times (0, T), \\
  y(0) = y_0 & \text{in } \Omega.
\end{cases}
\]  

The switching law \( \gamma = \gamma(t) \) is unknown. We need however to, at any time \( t \), make a choice of the control \( u = u(x, t) \), that may act either on \( \omega_1 \) or \( \omega_2 \), depending on the value of \( \gamma \), and this so that the final control requirement at time \( t = T \) is ensured.

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\(^1\)Q. Lu and E. Zuazua, Robust Null Controllability for Heat Equations with Unknown Switching Control Mode, DCDS, series B, to appear.
We consider the null controllability problem of system (13) which consists in driving the solution to rest,

$$y(x, T) \equiv 0,$$

(14)

for initial state $y_0 \in L^2(\Omega)$, by means of a suitable control $u$.

We emphasize that the control $u$ at time $t$ has to be chosen ignoring the switching pattern in future times $t \leq \tau \leq T$. 
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The following holds:

**Theorem**

There is a sequence \( \{ t_i \}_{i=1}^{\infty} \) with \( \lim_{i \to \infty} t_i = T \) and \( 0 = t_1 < t_2 < \cdots \) so that for every \( y_0 \in L^2(\Omega) \), we can find a control \( u(\cdot) \in L^\infty(0, T; L^2(\omega_1 \cup \omega_2)) \), such that

\[
u(t) = \begin{cases} 
\text{a function independent of } t, & \text{if } t \in (t_{2k-1}, t_{2k}), \ k \in \mathbb{N}, \\
0, & \text{if } t \in (t_{2k}, t_{2k+1}), \ k \in \mathbb{N},
\end{cases}
\]

which drives \( y \) to the rest at \( t = T \). Furthermore, there exists a constant \( L > 0 \) such that

\[
|u|_{L^\infty(0, T; L^2(\omega_1 \cup \omega_2))}^2 \leq L|y_0|_{L^2(\Omega)}^2 \quad (15)
\]

for all measurable switching functions \( \gamma(\cdot) : \mathbb{R} \to \{0, 1\} \) and all \( y_0 \) in \( L^2(\Omega) \).
The control $u$ depends on the past history of the switching function $\gamma$. This is reasonable, since possible variations of $\gamma$ modify the dynamics of the system.

But the control does not depend on the future unknown dynamics of the switching function.

In fact, we can show that the control can not be completely independent of $\gamma$. 
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Inspired on Lebeau-Robbiano’s\(^2\) strategy, together with the corresponding finite-dimensional analog.

LR’s argument exploits the possibility of controlling first \([0, T/2]\) and then relaxing the dynamics in \([0, T]\). This is combined with a dyadic decomposition, the fact that controlling the eigencomponents \(\{\lambda \leq N\}\) has a cost of the order of \(\exp(C\sqrt{N})\), and that the free dynamics for data whose projection on this eigenspace vanishes decays like \(\exp - Nt\).

The key fact is the following estimate on the linear independence of eigenfunctions: For suitable constants \(c, C > 0\),

\[
\int_\omega \left| \sum_{\lambda \leq N} a_k \phi_k(x) \right|^2 dx \geq C \exp(-c\sqrt{N}) \sum_{\lambda \leq N} a_k^2,
\]

and this for all \(N\) and \(\{a_k\}\).

---

This estimate allows limiting the analysis to finite-dimensional projections. Consider the following controlled finite-dimensional system.

\[
\begin{cases}
    z_t = A_m z + \gamma B_m^{(1)} f_1 + (1 - \gamma) B_m^{(2)} f_2 & \text{in } [t_1, t_2], \\
    z(t_1) = z_0.
\end{cases}
\] (16)

Here, \( f_1(\cdot) \) and \( f_2(\cdot) \) are controls taken from \( L^\infty(t_1, t_2; \mathbb{R}^m) \), \( z_0 \in \mathbb{R}^m \), and \( A_m = \text{diag}(-\lambda_1, \cdots, -\lambda_m) \) with \( 0 < \lambda_1 \leq \cdots \leq \lambda_m \),

\[
B_m^{(1)} = \left( \int_{G_1} e_i e_j dx \right)_{1 \leq i, j \leq m}, \quad B_m^{(2)} = \left( \int_{G_2} e_i e_j dx \right)_{1 \leq i, j \leq m}.
\]
Let $m \in \mathbb{N}$. Then for each $z_0 \in \mathbb{R}^m$, the controls $f_1(\cdot)$ and $f_2(\cdot)$ defined by

$$
\begin{align*}
f_1(t) &\equiv -(B_m^{(1)})^{-1}\left( \int_{t_1}^{t_2} e^{A_m(t_1-s)} ds \right)^{-1} z_0, \quad t \in (t_1, t_2), \\
f_2(t) &\equiv -(B_m^{(2)})^{-1}\left( \int_{t_1}^{t_2} e^{A_m(t_1-s)} ds \right)^{-1} z_0, \quad t \in (t_1, t_2),
\end{align*}
$$

drive the solution $z(\cdot; z_0, \gamma, f_1, f_2)$ from $z_0$ at time $t_1$ to the origin at time $t_2$. Furthermore, these controls satisfy

$$
\begin{align*}
|f_1|_{L^\infty(t_1,t_2;\mathbb{R}^m)}^2 &\leq C_1^2 e^{2C_1\sqrt{\lambda_m}} \left\| \left( \int_{t_1}^{t_2} e^{A_m(t_1-s)} ds \right)^{-1} z_0 \right\|_{\mathbb{R}^m}^2, \\
|f_2|_{L^\infty(t_1,t_2;\mathbb{R}^m)}^2 &\leq C_2^2 e^{2C_2\sqrt{\lambda_m}} \left\| \left( \int_{t_1}^{t_2} e^{A_m(t_1-s)} ds \right)^{-1} z_0 \right\|_{\mathbb{R}^m}^2.
\end{align*}
$$
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1. Investigate whether the controls we have obtained are optimal from a complexity viewpoint. For instance, are they those that switch a minimal number of times?

2. Is there a characterization by duality of the robustness control property we have proved? What is the corresponding observability inequality? Can it be obtained by means of the existing methods (Fourier series, Carleman inequalities, …)?

Our results yield the observability inequality

\[ |\varphi(0)|^2_{L^2(G)} \leq C \int_0^T \min \left\{ \int_{\omega_1} |\varphi(x, t)|^2 \, dx, \int_{\omega_2} |\varphi(x, t)|^2 \, dx \right\} \, dt, \quad (17) \]

for every \( \varphi(\cdot) \) solving

\[
\begin{cases}
\varphi_t + \Delta \varphi = 0 & \text{in } \Omega \times (0, T), \\
\varphi = 0 & \text{on } \partial \Omega \times (0, T), \\
\varphi(T) = \varphi_T & \text{in } \Omega.
\end{cases}
\]
We have seen that:

- Minimizing a functional of the form

\[
J_s(\varphi^0) = \frac{1}{2} \int_0^T |b \cdot \varphi(t)|^2 dt - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),
\]

leads to controls of minimal \(L^2(0, T)\)-norm.

- Minimizing a functional of the form

\[
J_s(\varphi^0) = \frac{1}{2} \left[ \int_0^T |b \cdot \varphi(t)| dt \right]^2 - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),
\]

leads to bang-bang controls of minimal \(L^\infty(0, T)\)-norm.

- Minimizing a functional of the form

\[
J_s(\varphi^0) = \frac{1}{2} \| b \cdot \varphi(t) \|_{L^\infty(0, T)}^2 - x^1 \cdot \varphi^0 + x^0 \cdot \varphi(0),
\]

leads to sparse controls of minimal \(L^1(0, T)\)-norm, linear combinations of Dirac masses on the set of maxima of minimizers.\(^3\)

\(^3\)E. Casas and E. Zuazua, Spike Controls for Elliptic and Parabolic PDE, SCL, 62 (2013), 311-318.
Outline

1. Switching control
   - Motivation
   - Finite-dimensional models
   - The $1-d$ heat equation
   - The multi-dimensional heat equation

2. Robust switching control of heat processes
   - Problem formulation
   - Main result
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4. Averaged control of uncertain systems
   - Motivation
   - Averaged control of ODE
   - Averaged control of PDEs
Often the data of the system under consideration or even the PDE (its parameters) describing the dynamics are not fully known. In those cases it is relevant to address control problems so to ensure that the control mechanisms:

- Are robust with respect to parameter variations.
- Guarantee a good control theoretical response of the system at least in an averaged sense.
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Consider the finite dimensional linear control system

\[
\begin{aligned}
    x'(t) &= A(\nu)x(t) + Bu(t), \quad 0 < t < T, \\
    x(0) &= x^0.
\end{aligned}
\]  

(21)

In (21) the (column) vector valued function 
\[x(t, \nu) = (x_1(t, \nu), \ldots, x_N(t, \nu)) \in \mathbb{R}^N\] is the state of the system, \(A(\nu)\) is a \(N \times N\)-matrix and \(u = u(t)\) is a \(M\)-component control vector in \(\mathbb{R}^M\), \(M \leq N\).

- The matrix \(A\) is assumed to depend on a parameter \(\nu\) in a continuous manner. To fix ideas we will assume that the parameter \(\nu\) ranges within the interval \((0, 1)\).

- Note however that the control operator \(B\) is independent of \(\nu\), the same as the initial datum \(x_0 \in \mathbb{R}^N\) to be controlled.
Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for a control $u$ such that the solution of (21) satisfies

$$\int_0^1 x(T, \nu) d\nu = x^1.$$  \hfill (22)

This concept of averaged controllability differs from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.

When $A$ is independent of the parameter $\nu$, controllable systems can be fully characterized in algebraic terms by the rank condition

$$\text{rank} \begin{bmatrix} B, AB, \ldots, A^{N-1}B \end{bmatrix} = N.$$  \hfill (23)
The following holds:

**Theorem**

*Theorem.* Averaged controllability holds if and only the following rank condition is satisfied:

\[
\text{rank} \left[ B, \int_0^1 [A(\nu)]d\nu B, \ldots, \int_0^1 [A(\nu)]^{N-1}d\nu B, \ldots \right] = N. \quad (24)
\]
The adjoint system depends also on the parameter \( \nu \):

\[
\begin{cases}
  -\varphi'(t) = A^*(\nu)\varphi(t), & t \in (0, T) \\
  \varphi(T) = \varphi^0.
\end{cases}
\] (25)

Note that, for all values of the parameter \( \nu \), we take the same datum for \( \varphi \) at \( t = T \). This is so because our analysis is limited to the problem of averaged controllability.

The corresponding averaged observability property reads:

\[
|\varphi^0|^2 \leq C \int_0^T \left| B^* \int_0^1 \varphi(t, \nu) d\nu \right|^2 dt.
\]
Since we are working in the finite-dimensional context, inequality (25) is equivalent to the following uniqueness property:

$$B^* \int_0^1 \varphi(t, \nu) d\nu = 0 \quad \forall t \in [0, T] \Rightarrow \varphi^0 \equiv 0. \quad (26)$$

To analyze this inequality we use the following representation of the adjoint state:

$$\varphi(t, \nu) = \exp[A^*(\nu)(T - t)]\varphi^0.$$

Then, the fact that

$$B^* \int_0^1 \varphi(t, \nu) d\nu = 0 \quad \forall t \in [0, T]$$

is equivalent to

$$B^* \int_0^1 \exp[A^*(\nu)(t - T)] d\nu \varphi^0 = 0 \quad \forall t \in [0, T].$$

The result follows using the time analyticity of the matrix exponentials, and the classical argument consisting in taking consecutive derivatives at time $t = T$. 
The simultaneous observability inequality reads

\[ |\varphi_1^0|^2 + |\varphi_2^0|^2 \leq C \int_0^T |B^* [\varphi_1 + \varphi_2]|^2 dt, \quad \forall \varphi_j^0 \in \mathbb{R}^N, \ j = 1, 2, \]  

(27)

For averaged controllability it is sufficient this to hold in the particular case where \( \varphi_1^0 = \varphi_2^0 \).
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The same analysis can be developed for systems of wave equations depending on parameters\textsuperscript{4}

Consider the following more complex system

\begin{align*}
  u_{1,t} - \Delta u_1 &= 0, \quad (t,x) \in \mathbb{R} \times \Omega \\
  u_{2,tt} - \Delta u_2 &= 0, \quad (t,x) \in \mathbb{R} \times \Omega \\
  u_i &= 0, \quad (t,x) \in \mathbb{R} \times \partial \Omega, \quad i = 1, 2 \\
  u_1(0, \cdot) &= \varphi(\cdot) \quad x \in \Omega.
\end{align*}

We assume that the main dynamics, the one we want to observe, is given by \( u_1 \), governed by the heat equation.

The wave solution, \( u_2 \), is then adding an additive perturbation.

Note that no information is given on \( u_2 \), other than being the solution of the wave equation. In particular, nothing is known on its initial data.

\textsuperscript{4}M. Lazar and E. Zuazua, Averaged control and observation of parameter-depending wave equations.
We know that there exists a constant $C$ such that the following estimate holds
\[
\sum_k e^{-c\sqrt{\lambda_k}} |\hat{\phi}_k|^2 \leq C \int_0^T \int_\omega |u_1|^2 \, dx \, dt.
\] (28)

We claim that
\[
\sum_k e^{-c\sqrt{\lambda_k}} |\hat{\phi}_k|^2 \leq C \int_0^T \int_\omega |u_1 + u_2|^2 \, dx \, dt,
\] (29)

for all solutions $(u_1, u_2)$ of the above system.
\( P_1 = \partial_t - \Delta \), \( P_2 = \partial_{tt} - \Delta \). Observe that

\[
\nu_1 = P_2(u_1 + u_2) = P_2 u_1. \tag{30}
\]

solves the heat equation with the same Dirichlet boundary conditions:

\[
\nu_{1,t} - \Delta \nu_1 = 0, \quad (t, x) \in \times \Omega, \quad \nu_1 = 0, \quad (t, x) \in \times \partial \Omega.
\]

Using

\[
\sum_k e^{-c\sqrt{\lambda_k} |\hat{\phi}_k|^2} \leq C_s \| \nu_1 \|_{H^{-s}(\omega \times (0, T))}^2. \tag{31}
\]

we get

\[
\| \nu_1 \|_{H^{-2}(\omega \times (0, T))} = \| P_2(u_1) \|_{H^{-2}(\omega \times (0, T))} = \| P_2(u_1 + u_2) \|_{H^{-2}(\omega \times (0, T))} \leq C \| u_1 + u_2 \|_{L^2(\omega \times (0, T))}.
\]
Open problems

- Note that, in particular, these arguments cannot be applied for systems involving an infinite number of equations.
- The arguments get much more involved when the PDE under consideration have variable coefficients.