Sharp observability estimates for heat equations

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THE GENERAL PROBLEM: NULL CONTROL OR CONTROL TO ZERO

TO CONTROL TO THE NULL EQUILIBRIUM STATE PARABOLIC EQUATIONS BY MEANS OF A CONTROL (RIGHT HAND SIDE TERM) CONCENTRATED ON AN OPEN SUBSET OF THE DOMAIN WHERE THE EQUATION HOLDS.

EQUIVALENT FORMULATION: OBSERVABILITY

ANALYZE HOW MUCH OF THE TOTAL ENERGY OF SOLUTIONS CAN BE OBTAINED OUT OF LOCAL MEASUREMENTS.

OBSERVATION $\equiv$ CONTROL

THE CONTROL PROBLEM

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits an unique solution

$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

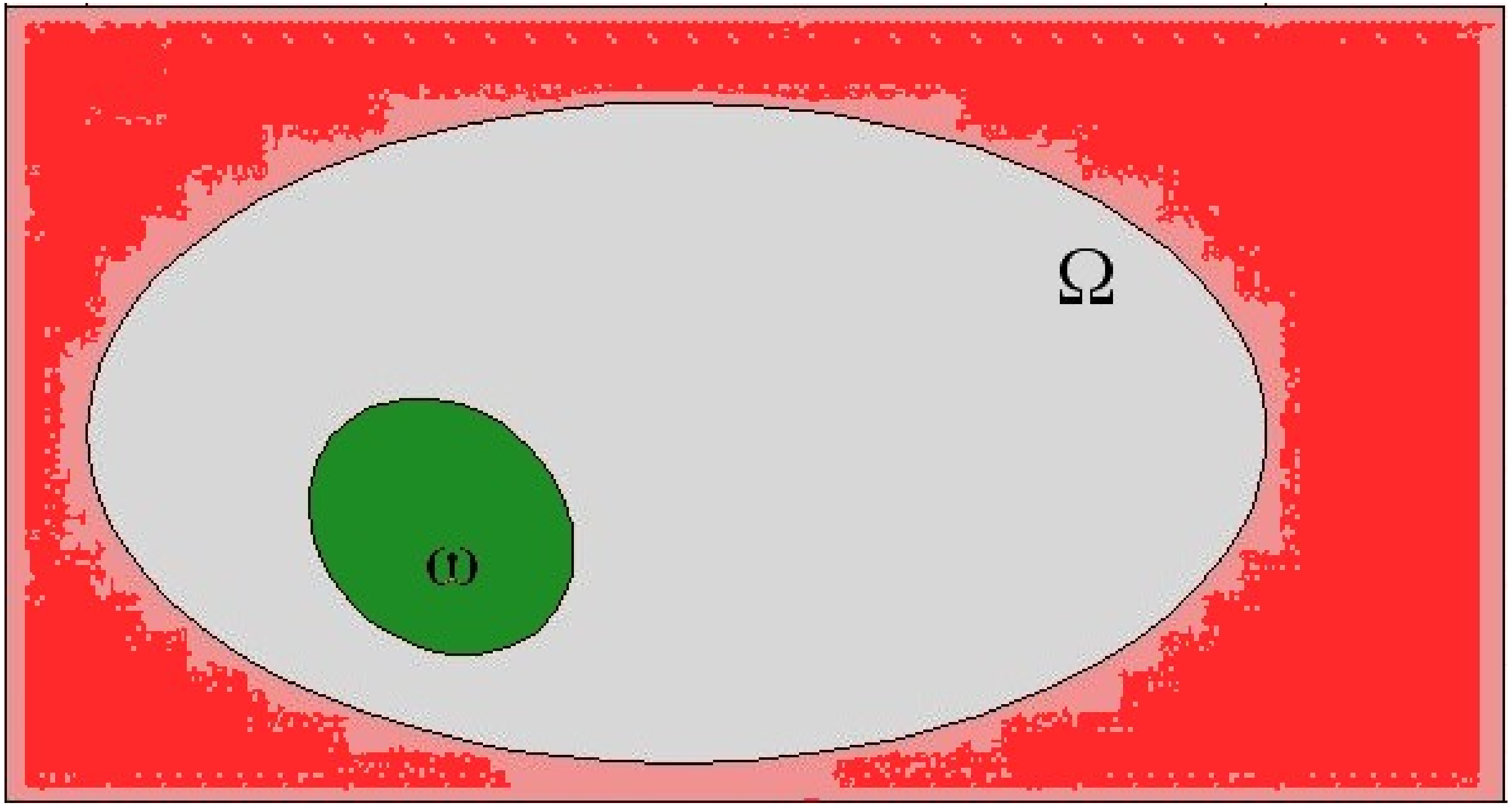
$$u = u(x,t) = solution = state, \; f = f(x,t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function f.

We introduce the reachable set $R(T; u^0) = \left\{ u(T) : f \in L^2(Q) \right\}$.

Approximate controllability: $R(T; u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

Null controllability: if $0 \in R(T; u^0)$ for all $u^0 \in L^2(\Omega)$.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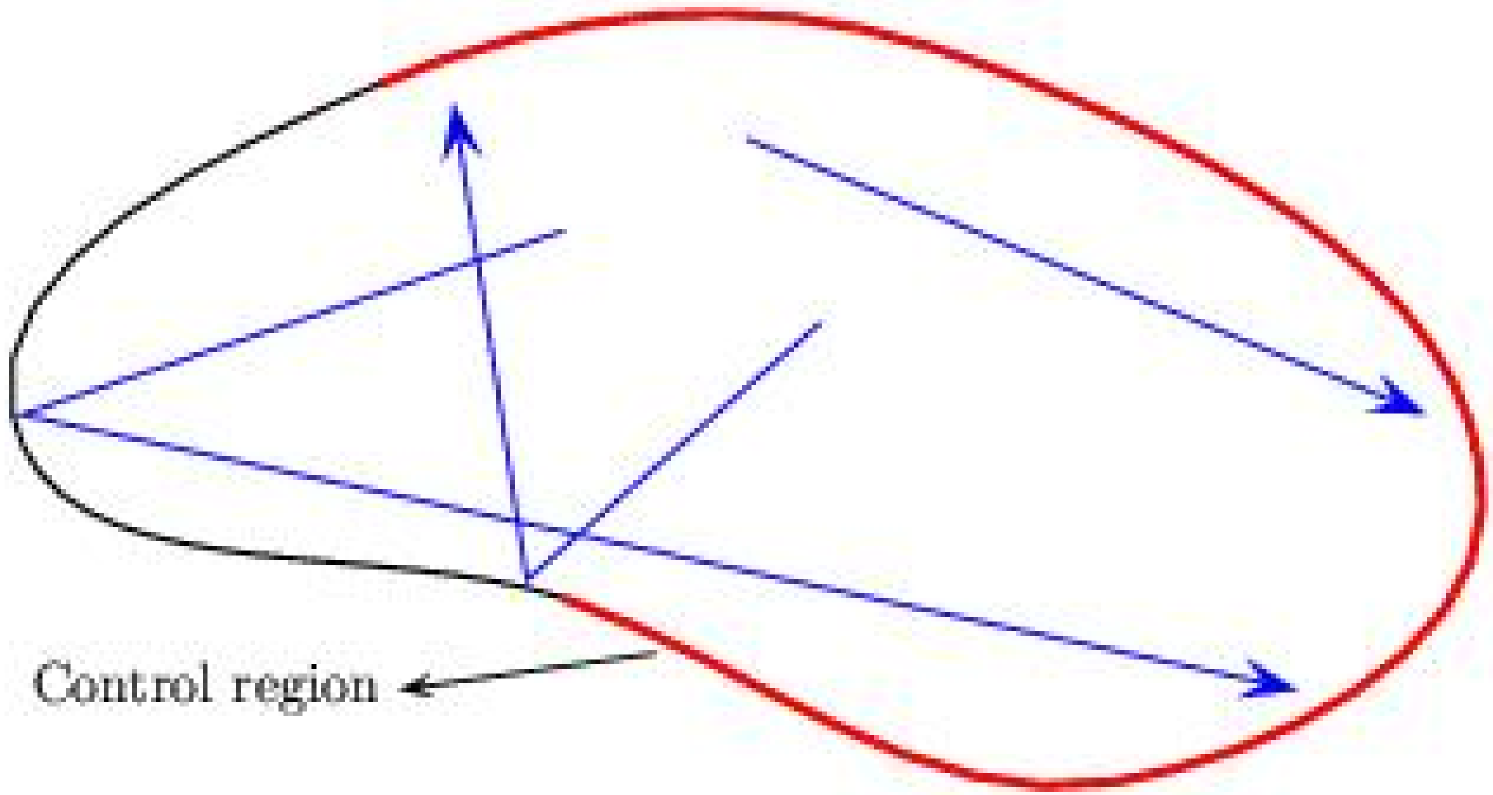
In principle, due to the intrinsic infinite velocity of propagation of the heat equation, one can not exclude these properties to hold in any time $T > 0$ and from any open non-empty open subset ω of Ω.

Note that for similar properties to hold for wave equations, typically, one needs to impose geometric conditions on the control subset and the time of control, namely, the so called GCC (Geometric Control Condition) by Bardos-Lebeau-Rauch: It asserts, roughly, that all rays of geometric optics enter the control set ω in time T.



But this kind of Geometric Condition in unnecessary for the heat equation.

Approximate controllability

For all initial data u^0, all final data $u^1 \in L^2(\Omega)$ and all $\varepsilon > 0$ there exists a control f_ε such that the solution satisfies:

$$||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon.$$

- Approximate controllability does not guarantee that the target u^1 may be reached exactly. It could well be that $||f_\varepsilon||_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$.

- The property as such is of little use in practice since too large controls might be impossible to implement.

Approximate controllability is in fact equivalent to an unique continuation property for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2}$$

More precisely, approximate controllability holds if and only if the following uniqueness or unique continuation property (UCP) is true:

$$\varphi = 0 \text{ in } \omega \times (0,T) \implies \varphi \equiv 0, \text{ i.e. } \varphi^0 \equiv 0. \tag{3}$$

This UCP is a consequence of Holmgren's uniqueness Theorem.

This is so for all ω and all $T > 0$.

UCP $\Longrightarrow$ Approximate controllability

Consider the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega \varphi^2 dxdt + \varepsilon \left|\varphi^0\right|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx + \int_\Omega \varphi(0) u^0 dx. \quad (4)$$

$J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ is continuous, and convex.

Moreover, UCP implies coercivity:

$$\lim_{||\varphi^0||_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(\varphi^0)}{||\varphi^0||_{L^2(\Omega)}} \geq \varepsilon.$$

Accordingly, the minimizer $\widehat{\varphi}^0$ exists and the control

$$f_\varepsilon = \widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of the adjoint system corresponding to the minimizer is the control such that

$$||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon.$$

This is a general principle:

UCP $\Longrightarrow$ APPROXIMATE CONTROLLABILITY

Moreover, there is a variational characterization of the optimal control.

This argument does not provide any estimate on the size of the control f_ε as $\varepsilon \to 0$. Roughly speaking:

- For a very narrow set of exactly reachable u^1 states the controls f_ε are bounded and converge as $\varepsilon \to 0$ to a control f such that

  $$u(T) = u^1.$$

  This necessarily happens for a small class of u^1 because of the regularizing effect of the heat equation.

- Typically, for targets u^1 which are in a Sobolev class, the controls f_ε diverge exponentially on $1/\varepsilon^\alpha$, for some α depending on the Sobolev class they belong to.

Null controllability

For achieving

$$u(T) = 0$$

we have to consider the case in which

$$u^1 = 0, \, \varepsilon = 0.$$

Thus, we are led to considering the functional

$$J_0(\varphi^0) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx \qquad (5)$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$.

Is it coercive?

For coercivity the following observability inequality is needed:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{6}$$

This inequality is very likely to hold: because of the very strong regularizing effect of the heat equation the norm of $\varphi(0)$ is a very weak measure of the total size of solutions. Indeed, in a Fourier series representation, the norm of $\varphi(0)$ presents weights which are of the order of $\exp(-\lambda_j T)$, $\lambda_j \to \infty$ being the eigenvalues of the Dirichlet $-\Delta$.

For the wave equation this observability inequality requires of the GCC (sufficiently large time and geometric conditions on the subset ω to absorbe all rays of Geometric Optcs). But for the heat equation there is no reason to think on the need of any restriction on T or ω.

Actually, this estimate was proved by Fursikov and Imanuvilov (1996) using Carleman inequalities. In fact the same proof applies for equations with smooth (C^1) variable coefficients in the principal part and for heat equations with lower order potentials.

Consider the heat equation or system with a potential $a = a(t, x)$ in $L^\infty(Q; \mathbb{R}^{N\times N})$:

$$\begin{cases} \varphi_t - \Delta\varphi + a\varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma, \\ \varphi(0, x) = \varphi^0(x), & \text{in } \Omega, \end{cases} \tag{7}$$

where φ takes values in $\mathbb{R}^N$.

Note that we have reversed the sense of time to make the inequality more intuitive and better underline the effect of the heat equation as time evolves: regularizing effect and possible exponential increase on the size of the solution due to the presense of the potential as Gronwall's inequality predicts.

Theorem A

(Fursikov+Imanuvilov, 1996, E. Fernández-Cara+E. Zuazua, 2000)

Assume that ω *is an open non-empty subset of* Ω*. Then, there exists a constant* $C = C(\Omega, \omega) > 0$*, depending on* Ω *and* ω *but independent of* T*, the potential* $a = a(t, x)$ *and the solution* φ *of* (29)*, such that*

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3} \right)\right) \int_0^T \int_\omega |\varphi|^2 dx dt, \tag{8}$$

for every solution φ *of* (29)*, potential* $a \in L^\infty(Q; \mathbb{R}^{N\times N})$ *and time* $T > 0$*.*

NOTE THAT IN THIS ESTIMATE NO INFORMATION ON THE INITIAL DATA IS BEING USED.

THUS, WE ARE DEALING WITH AN ILL-POSED PROBLEM IN WHICH "CAUCHY" DATA ARE ONLY GIVEN IN ω.

WE HAVE HOWEVER THE DIRICHLET B. C. EVERYWHERE ON THE BOUNDARY.

NOTE THAT THE ESTIMATES WE OBTAIN ARE INDEPENDENT OF THE SOLUTION. IN THIS SENSE THE RESULT IS BETTER THAN THE CLASSICAL ENERGY ESTIMATES FOR THE BACKWARD HEAT EQUATION THAT DEPEND ON THE WAVE NUMBER OF THE SOLUTION.

Sketch of the proof:

Introduce a function $\eta^0 = \eta^0(x)$ such that:

$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) & \\ \eta^0 > 0 & \text{in} \quad \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla\eta^0 \neq 0 & \text{in} \quad \overline{\Omega\backslash\omega}. \end{cases} \tag{9}$$

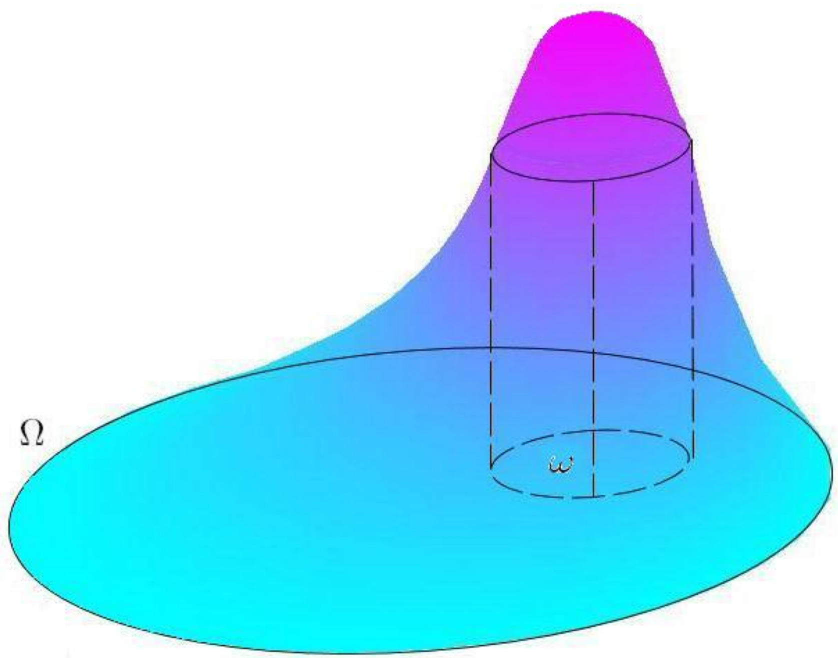
In some particular cases, for instance when Ω is star-shaped with respect to a point in ω, it can be built explicitly without difficulty. But the existence of this function is less obvious in general, when the domain has holes or its boundary oscillates, for instance.

Let $k > 0$ such that $k \geq 5\max_{\bar{\Omega}} \eta^0 - 6\min_{\bar{\Omega}} \eta^0$ and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4}\max \beta^0,\ \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t)).$$



The following Carleman inequality holds:

Proposition 1 *(Fursikov + Imanuvilov, 1996)*

There exist positive constants $C_*, s_1 > 0$ *such that*

$$\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \tag{10}$$

$$+s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1}|\nabla q|^2 dxdt + s^3\int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt$$

$$\leq C_*\left[\int_Q \rho^{-2s}|\partial_t q - \Delta q|^2 dxdt + s^3\int_0^T\int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt\right]$$

for all $q \in Z$ *and* $s \geq s_1$.

Moreover, C_* *depends only on* Ω *and* ω *and* s_1 *is of the form*

$$s_1 = s_0(\Omega, \omega)(T + T^2).$$

Let us go back to the estimate:

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1+\frac{1}{T}+T \| a \|_\infty + \| a \|_\infty^{2/3}\right)\right)\int_0^T\int_\omega |\varphi|^2 dxdt. \tag{11}$$

Three different terms have to be distinguished on the observability constant on the right hand side:

$$C(T,a) = C_1^*(T,a)C_2^*(T,a)C_3^*(T,a), \tag{12}$$

where

$$C_1^*(T,a) = \exp\left(C\left(1+\frac{1}{T}\right)\right), \quad C_2^*(T,a) = \exp(CT \| a \|_\infty), \tag{13}$$

$$C_3^*(T,a) = \exp\left(C \| a \|_\infty^{2/3}\right).$$

The role of the first two constants is clear:

• The first one $C_1^*(T, a) = \exp\left(C\left(1 + \frac{1}{T}\right)\right)$ takes into account the increasing cost of making continuous observations as T diminishes.

• The second one $C_2^*(T, a) = \exp(CT \parallel a \parallel_\infty)$ is due to the use of Gronwall's inequality to pass from a global estimate in (x, t) into an estimate for $t = T$.

What about the third one?

• $2/3 \in [1/2, 1]$!!!!!!!!!!!!!!: The exponent 1 is natural because we are dealing with a first-order (in time) equation. The exponent 2 is natural as well: We are propagating information on the x-direction. For that the governing operator is $-\Delta$ which is second order.

Theorem 1 *(Th. Duyckaerts, X. Zhang and E. Z., 2005)*

The third constant $C_3^*(T, a)$ is sharp in the range

$$\| a \|_\infty^{-2/3} \lesssim T \lesssim \| a \|_\infty^{-1/3}, \tag{14}$$

for systems $N \geq 2$ and in more than one dimension $n \geq 1$.

More precisely, there exists $c > 0$, $\mu > 0$, a family $(a_R)_{R>0}$ of matrix-valued potentials such that

$$\| a_R \| \underset{R \to +\infty}{\longrightarrow} +\infty,$$

and a family $(\varphi_R^0)_{R>0}$ of initial conditions in $(L^2(\Omega))^N$ so that the corresponding solutions φ_R with $a = a_R$ satisfy

$$\lim_{R\to\infty}\left\{\inf_{T\in I_\mu}\frac{\| \varphi_R(0) \|^2_{(L^2(\Omega))^N}}{\exp(c \| a_R \|_\infty^{2/3}) \int_0^T \int_\omega |\varphi_R|^2 dxdt}\right\} = +\infty. \tag{15}$$

where $I_\mu \triangleq \left(0, \mu \| a_R \|^{-1/3}\right]$.

Open problem: Optimality for scalar equations ($N = 1$) and in one space dimension ($n = 1$).

The proof is based on the following Theorem by V. Z. Meshkov, 1991.

Theorem 2 (Meshkov, 1991). *Assume that the space dimension is* $n = 2$. *Then, there exists a nonzero complex-valued bounded potential* $q = q(x)$ *and a non-trivial complex valued solution* $u = u(x)$ *of*

$$\Delta u = q(x)u, \qquad \textit{in } \mathbb{R}^2, \tag{16}$$

with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall\, x \in \mathbb{R}^2 \tag{17}$$

for some positive constant $C > 0$.

Remark 1

- *The growth rate* $\exp(- \mid x \mid^{4/3})$ *is optimal. Indeed, as proved by Meshkov using a Carleman inequality, if the solution decays faster it has to be zero. This is true for all* n *(space dimension) and* N *(size of the elliptic system):*

$$\forall v \in C_0^\infty(\{r > 1\}),$$
$$\tau^3 \int |v|^2 \exp(2\tau r^{4/3}) r^{2-n} dx \leq C \int |\Delta v|^2 \exp(2\tau r^{4/3}) r^{2-n} dx. \quad (18)$$

- *Constructing solutions decaying as* $\exp(- \mid x \mid^{4/3})$ *for scalar equations is an interesting open problem. The construction by Meshkov is based on a decomposition of* $\mathbb{R}^n$ *into concentric and divergent annulae in which the frequency of oscillation of harmonics* *increases and, simultaneously, the modulus of the solution diminishes. For doing that the particular structure of the spherical harmonics* $r^{-k}\exp(-ik\theta)$ *and, in particular, the fact that* $|\exp(-ik\theta)| = 1$ *plays a key role.*

- *We have extended this construction to* $3-d$ *but with a slightly weaker decay rate (polynomial loss). According to this we can prove that the observability estimate for the heat equation is almost sharp to, up to a logarithmic factor. Strict optimality in* $3-d$ *is open.*

$$\lim_{R\to\infty}\left\{\frac{\| w_R^0 \|^2_{(L^2(\Omega))^N} + \| w_R^1 \|^2_{(H^{-1}(\Omega))^N}}{\exp\left(c(\log \| a_R \|)^{-2} \| a_R \|^{2/3}\right)\int_0^T\int_\omega |w_R|^2 dxdt}\right\} = +\infty. \tag{19}$$

This is based on the following variant of Meshkov's elliptic construction:

$$q(x)(\log(2+r))^{-3} \in L^{\infty} \quad |u(x)| \leq Ce^{-r^{4/3}}. \tag{20}$$

The construction is based on a sequence of eigenfunctions of the Laplace-Beltrami operator for which quotients can be bounded above and below polynomially:

$$\forall \omega \in M, \quad \frac{1}{Cn_k^N} \leq \frac{|\Phi_k(\omega)|}{|\Phi_{k+1}(\omega)|} \leq Cn_k^N.$$

This sequence plays the role of $\exp(ik\theta)$ *in* $2-d$.

- *Using separation of variables these constructions (Meshkov for $n = 2$ and DZZ for $n = 3$) can be extended to an arbitrary number of space dimensions.*

- *In $1 - d$ an ODE argument shows that the decay rate is at most exponential. Thus, the superexponential decay for the elliptic problem can not be obtained and the optimality of the parabolic observability inequality can not be proved in this way.*

Sketch of proof

Step 1: Construction on $\mathbb{R}^n$.

Consider the solution u and potential q given by Meshkov's Theorem. By setting

$$u_R(x) = u(Rx), \qquad a_R(x) = R^2 q(Rx), \tag{21}$$

we obtain a one-parameter family of potentials $\{a_R\}_{R>0}$ and solutions $\{u_R\}_{R>0}$ satisfying

$$\Delta u_R = a_R(x) u_R, \qquad \text{in } \mathbb{R}^n \tag{22}$$

and

$$| u_R(x) | \leq C \exp\left(-R^{4/3} | x |^{4/3}\right), \qquad \text{in } \mathbb{R}^n. \tag{23}$$

These functions may also be viewed as stationary solutions of the corresponding parabolic systems. Indeed, $\psi_R(t,x) = u_R(x)$, satisfies

$$\psi_{R,t} - \Delta\psi_R + a_R\psi_R = 0, \qquad x \in \mathbb{R}^n,\ t > 0 \tag{24}$$

and

$$| \psi_R(x,t) | \leq C \exp(-R^{4/3} \mid x \mid^{4/3}), \qquad x \in \mathbb{R}^n,\ t > 0. \tag{25}$$

Step 2: Restriction to Ω.

Let us now consider the case of a bounded domain Ω and ω to be a non-empty open subset Ω such that $\omega \neq \Omega$. Without loss of generality (by translation and scaling) we can assume that $B \subset \Omega \backslash \bar{\omega}$.

We can then view the functions $\{\psi_R\}_{R>0}$ above as a family of solutions of the Dirichlet problem in Ω with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} \psi_{R,t} - \Delta\psi_R + a_R\psi_R = 0, & \text{in } Q, \\ \psi_R = \varepsilon_R, & \text{on } \Sigma, \end{cases} \tag{26}$$

where $\varepsilon_R = \psi_R\Big|_{\partial\Omega} = u_R\Big|_{\partial\Omega}$.

Taking into account that both ω and $\partial\Omega \subset B^c$ for a suitable C:

$$\mid \psi_R(t,x) \mid \leq C \exp\left(-R^{4/3}\right), \quad x \in \omega,\ 0 < t < T,$$

$$\mid \varepsilon_R(t,x) \mid \leq C \exp\left(-R^{4/3}\right), \quad x \in \partial\Omega,\ 0 < t < T$$

$$\| \psi_R(T) \|^2_{L^2(\Omega)} \sim \| \psi_R(T) \|^2_{L^2(\mathbb{R}^n)} = \| u_R \|^2_{L^2(\mathbb{R}^n)} = \frac{1}{R^n} \| u \|^2_{L^2(\mathbb{R}^n)} = \frac{c}{R^n}$$

$$\| a_R \|_{L^\infty(\Omega)} \sim \| a_R \|_{L^\infty(\mathbb{R}^n)} = CR^2.$$

We can then correct these solutions to fulfill the Dirichlet homogeneous boundary condition. For this purpose, we introduce the correcting terms

$$\begin{cases} \rho_{R,t} - \Delta\rho_R + a_R\rho_R = 0, & \text{in } Q, \\ \rho_R = \varepsilon_R, & \text{on } \Sigma, \\ \rho_R(0,x) = 0, & \text{in } \Omega, \end{cases} \tag{27}$$

and then set

$$\varphi_R = \psi_R - \rho_R. \tag{28}$$

Clearly $\{\varphi_R\}_{R>0}$ is a family of solutions of parabolic systems with potentials $a_R(x) = R^2 q(Rx)$.

The exponential smallness of the Dirichlet data ε_R shows that ρ_R is exponentially small too. This allows showing that φ_R satisfies essentially the same properties as ψ_R. Thus, the family φ_R suffices to show that the optimality of the 2/3-observability estimate for the heat equation.

$1-d$ POTENTIALS DEPENDING ONLY ON x

Consider first the wave equation

$$u_{tt} - u_{xx} + a(x)u = 0.$$

Using sidewise energy estimates (the fact that the equation is also well-posed in the sense of x-this is so because it remains to be a wave equation when reversing the sense of x and t) we get an observability estimate of the form

$$C(a) = \exp\left(c(T)\sqrt{||a||_\infty}\right).$$

Once more the key ingredient is that the wave equation is second order in x and a carefull application of Gronwall's lemma yields an exponential factor on $\sqrt{||a||}$ instead of $||a||$.

We can then use Kannai's transform to write solutions of the heat equation

$$\varphi_t - \varphi_{xx} + a(x)\varphi = 0,$$

in terms of the solution of the wave equation.

This means that, for this heat equation, with potential depending only on x (!!!!!) the observability estimate is rather of the form:
$\| \varphi(T) \|^2_{(L^2(\Omega))^N}$
$\leq \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + C(T) \| a \|_\infty^{1/2}\right)\right) \int_0^T \int_\omega |\varphi|^2 dxdt,$

What about the case where $a = a(x,t)$?.

EQUATIONS WITH CONVECTIVE POTENTIALS

We may also consider equations or systems with convective potentials of the form

$$\begin{cases} \varphi_t - \Delta\varphi + W\cdot\nabla\varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x), & \text{in } \Omega, \end{cases} \tag{29}$$

For these equations the observability inequality reads:

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N}$$
$$\leq \exp\left(C\left(1+\frac{1}{T}+T\,\|\,W\,\|_\infty + \|\,W\,\|^2_\infty\right)\right)\int_0^T\int_\omega |\varphi|^2 dxdt.$$

Note that the estimate depends exponentially quadratically on the potential. The estimate is sharp in the class of multi-dimensional systems. The main ingredient is Meshkov's construction. A careful analysis shows that the underlying elliptic equation is:

$$-\Delta u = W(x) \cdot \nabla u$$

with the same u decaying as $\exp(-|x|^{4/3})$ and the potential $W(x)$ such that

$$(|x| + 1)^{1/3} |W(x)| \leq C.$$

Similar open problems arise for scalar equations and in one space dimension.

OBSERVABILITY and GEOMETRY

In the absence of potential, the Carleman inequality yields the following observability estimate for the solutions of the heat equation:

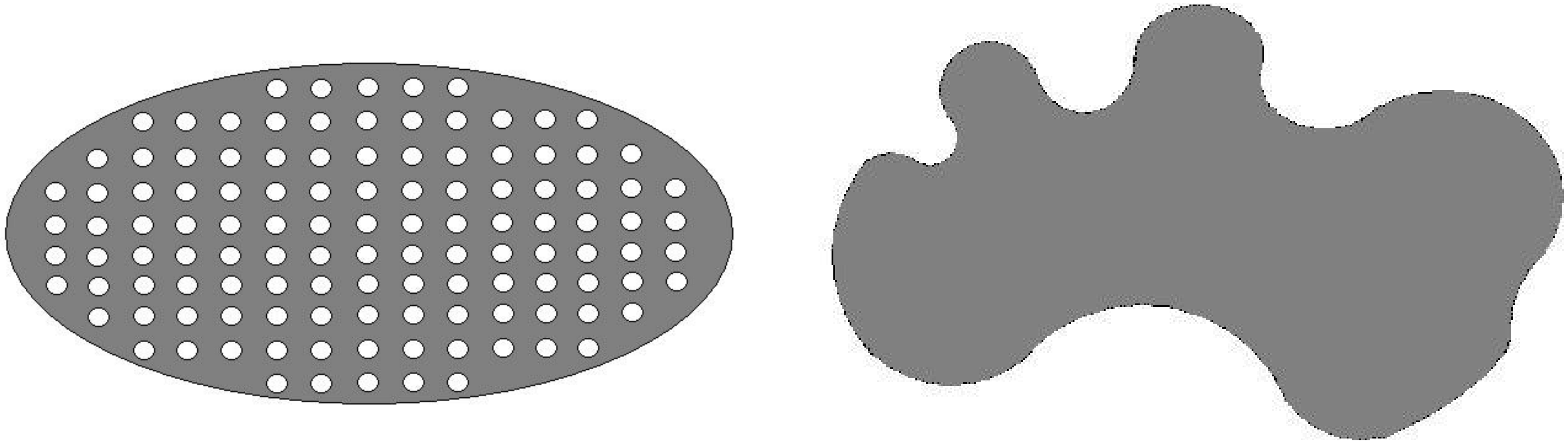
$$\int_0^\infty \int_\Omega e^{\frac{-A}{t}} \varphi^2 dx dt \leq C \int_0^\infty \int_\omega \varphi^2 dx dt.$$

Open problem: Characterize the best constant A in this inequality:

$$A = A(\Omega, \omega).$$

- The Carleman inequality approach allows establishing some upper bounds on A depending on the properties of the weight function.

But this does not give a clear path towards the obtention of a sharp constant.



- By inspection of the heat kernel one can see that for the inequality to be true one needs

$$A > \ell^2/2$$

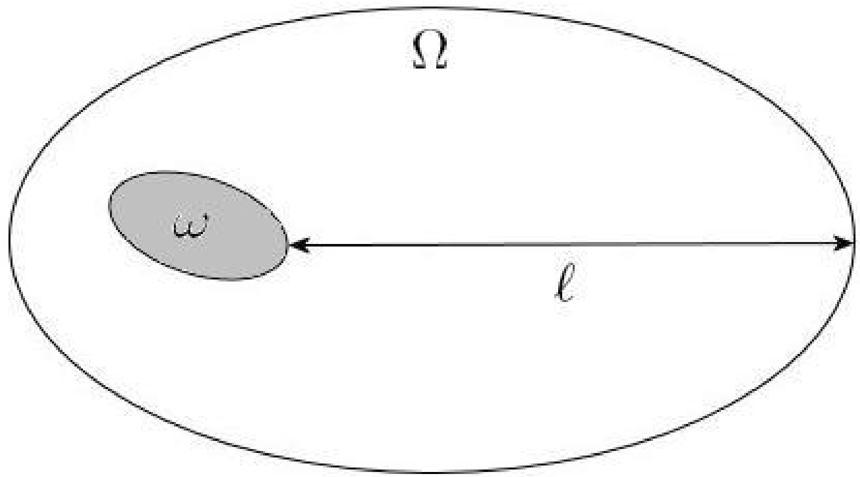
where ℓ is the length of the largest geodesic in $\Omega \setminus \omega$.

This was done by L. Miller (2003) using Varadhan's formula for the bhevaior of the heat kernel in short times and can also be done by the upper bounds on the Green function (see Davies' book). Recall that, the heat kernel is given by, Recall that:

$$G(x,t) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right).$$

then, the following upper bound holds for the Green function in Ω:

$$G_\Omega(x,y,t) \leq Ct^{-n/2} \exp\left(\frac{-d^2(x,y)}{(4+\delta)t}\right).$$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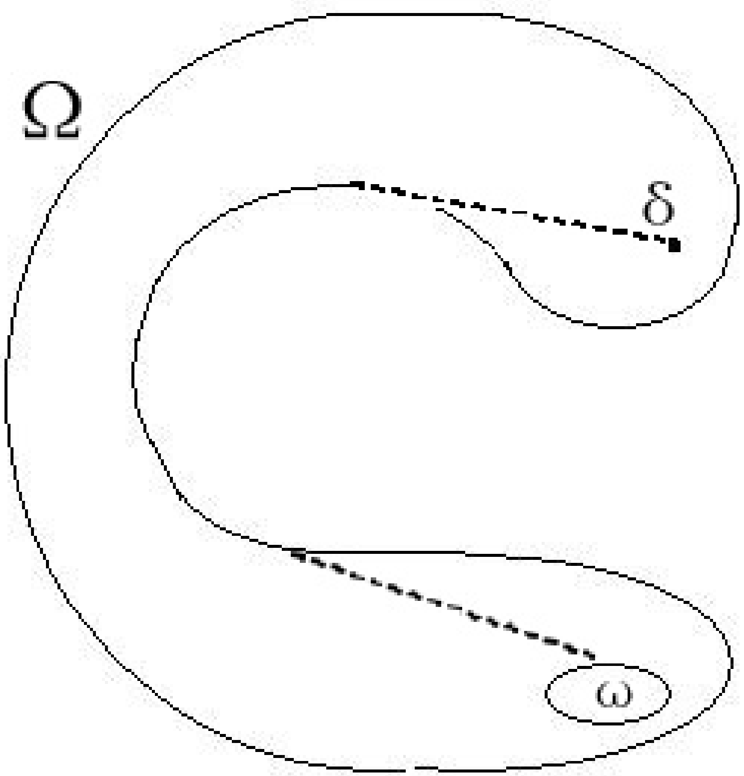

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The spectral approach

Lebeau and Robbiano proposed (1996) a spectral proof of the null controllability that, by duality, yields observability inequalities too. The key ingredient is the following estimate on the linear independence of restrictions of eigenfunctions of the laplacian:

Theorem 3 *(*Lebeau + Robbiano, 1996)

Let Ω be a bounded domain of class C^∞. For any non-empty open subset ω of Ω there exist $B, C > 0$ such that

$$Ce^{-B\sqrt{\mu}} \sum_{\lambda_j \leq \mu} | a_j |^2 \leq \int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \qquad (30)$$

for all $\{a_j\} \in \ell^2$ and for all $\mu > 0$.

Geometric open problem: To characterize the best constant $B = B(\Omega, \omega)$.

Is the constant B in this spectral inequality related to the best constant $A > 0$ in the parabolic one?

By inspection of the gaussian heat kernel it can be shown that this estimate, i. e. the degeneracy of the constant as $\exp(-B\sqrt{\mu})$ for some $B > 0$, is sharp even in $1-d$.

Although the constant $Ce^{-B\sqrt{\mu}}$ degenerates exponentially as $\mu \to \infty$, it is important that it does it exponentially on $\sqrt{\mu}$. The strong dissipativity of the heat equation allows compensating this degeneracy and to control the system, after all.

- As a consequence of the spectral estimate one can prove that the observability inequality holds for solutions with initial data in $E_\mu = \text{span}\left\{\psi_j\right\}_{\lambda_j \leq \mu}$, the constant being of the order of $\exp\left(B\sqrt{\mu}\right)$. This shows that the projection of solutions over E_μ can be controlled to zero with a control of size $\exp\left(B\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \leq \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(B\sqrt{\mu}\right)$.

  This holds in fact for all evolution PDE allowing a Fourier decomposition on the basis of the eigenfunctions of the laplacian.

- However, solutions of the heat equation without control ($f = 0$) and such that the projection of the initial data over E_μ vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$. This can be easily seen by means of the Fourier series decomposition of the solution.

Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \leq \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0:

$$\exp(B\sqrt{\mu})\exp(-T\mu/2) << 1.$$

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution.

– To control the whole solution an iterative argument is needed in which the interval $[0, T]$ has to be decomposed in a suitably chosen sequence of subintervals $[T_k, T_{k+1})$ and the argument above is applied in each subinterval to control an increasing range of frequencies $\lambda \leq \mu_k$ with $\mu_k \to \infty$ at a suitable rate.

When the evolution equation under consideration allows a Fourier series decomposition this argument is extremely useful. For instance it allows controlling equations of the form

$$y_t + (-\Delta y)^{\alpha} = 0,$$

with $\alpha > 1/2$.

On the other hand, the null control property fails for $\alpha = 1/2$ (S. Micu & E. Z, 2003):

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j} = \infty, \; \mu_j = j.$$

This shows that this iterative construction provides sharp results.

THE WAVE EQUATION

Consider now the wave equation:

$$\begin{cases} w_{tt} - \Delta w + aw = 0, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0,x) = w^0(x), \quad w_t(0,x) = w^1(x), & \text{in } \Omega. \end{cases} \tag{31}$$

Let us discuss the observability estimate:

$$\| w^0 \|^2_{(L^2(\Omega))^N} + \| w^1 \|^2_{(H^{-1}(\Omega))^N} \leq D^*(T,a) \int_0^T \int_\omega |w|^2 dxdt, \tag{32}$$

In this case, for this to be true, the GCC is needed.

We assume for instance that ω is a neighborhood of the boundary and that $T > 0$ is large enough.

The following holds, as a consequence of Carleman inequalities:

$$D^*(T,a) \leq \exp\left(C(T)\left(1+ \parallel a \parallel_\infty^{2/3}\right)\right), \qquad (33)$$

According to Meshkov's example this estimate is sharp too in dimensions $n \geq 2$. But it is not optimal in $1-d$ since sidewise energy estimates allow showing an estimate where the term $||a||^{2/3}$ can be replaced by $||a||^{1/2}$.

It is important to note some differences with respect to the heat equation: The linear term $T||a||$ on the exponential factor does not appear this time. This is due to the fact that, the wave equation being of order two in time, a more careful version of Gronwall's Lemma provides a growth estimate of the energy of the order of $\exp(T||a||^{1/2})$. Obviously, this term can be bounded above by $\exp\left(C(T)\left(1+\parallel a\parallel_{\infty}^{2/3}\right)\right)$.

CONSEQUENCES ON THE CONTROL OF NONLINEAR SYSTEMS

Consider semilinear parabolic equation of the form

$$\begin{cases} y_t - \Delta y + g(y) = f 1_\omega & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{34}$$

Theorem 4 *(E. Fernández-Cara + EZ, Annales IHP, 2000) The semilinear system is null controllable if*

$$g(s)/\mid s \mid \log^{3/2} \mid s \mid \rightarrow 0 \text{ as } \mid s \mid \rightarrow \infty. \tag{35}$$

Note that blow-up phenomena occur if

$$g(s) \sim \mid s \mid \log^p(1+ \mid s \mid), \text{ as } \mid s \mid \rightarrow \infty$$

with $p > 1$. Thus, in particular, weakly blowing-up equations may be controlled.

On the other hand, it is also well known that blow-up may not be avoided when $p > 2$ and then control fails.

Note that in the control process the propagation of energy in the x direction plays a key role. When viewing the underlying elliptic problem $\Delta y + g(y)$ a a second order differential equation in x we see how the critical exponent $p = 2$ arises. For $p > 2$ concentration in space may occur so that the control may not avoid the blow-up to occur outside the control region ω.

Sketch of the proof. Linearization + fixed point.

We linearize the system

$$\begin{cases} y_t - \Delta y + h(z)y = f 1_\omega & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega, \end{cases} \tag{36}$$

with

$$h(z) = g(z)/z.$$

Note that, if $z = y$, $h(z)y = g(y)$. In that case solutions of the linearized system are also solutions of the semilinear one.

The cost of controlling the system is of the form:

$$||f|| \leq ||y^0|| \exp\left(C\left(1 + \frac{1}{T} + T \parallel g(z) \parallel_\infty + \parallel g(z) \parallel_\infty^{2/3}\right)\right).$$

But $g(z) \sim \log^p(z)$. Thus

$$||f|| \leq ||y^0|| \exp\left(C\left(1 + \frac{1}{T} + T\log^p(||z||) + \log^{2p/3}(||z||)\right)\right).$$

When $||z||$ is large the term $\log^p(||z||)$ dominates but can be compensated by taking T small enough:

$$T\log^p(||z||) \sim \log^{2p/3}(||z||)$$

i.e.

$$T \sim \log^{-p/3}(||z||).$$

In this way

$$||f|| \leq ||y^0|| \exp\left(C\left(1+\frac{1}{T}+\log^{2p/3}(||z||)\right)\right) \sim C||y^0|| \exp\left(C\log^{2p/3}(||z||)\right).$$

Obviously, if

$$2p/3 < 1$$

this yields a sublinear estimate: with $0 < \gamma < 1$,

$$||f|| \leq C||y^0||||z||^\gamma.$$

This allows also proving a sublinear estimate for the map $z \to y$. Schauder's fixed point Theorem can be applied. A fixed point for which $z = y$ exists. This fixed point solves the semilinear equation and satisfies the final requirement $y(T) = 0$.

Underlying idea: Take T small, control quickly, before blow-up occurs.

As we have said, this argument can not be applied for $p > 2$.

What happens for $3/2 < p < 2$?

Our analysis of the optimality of linear observability estimates shows that this fixed point method can not do better. Is the equation null controllable or not in that range?

For wave equations, the observability inequality we have obtained allows controlling wave equations with nonlinearities growing as $g(s) \sim s \log^{3/2}(s)$. But, the velocity of propagation being finite the wave equation may not be controlled in the presence of blow-up phenomena. Both results are compatible since, the wave equation being second order in time, blow-up may only occur for nonlinearities of the form $g(s) \sim s \log^{p}(s)$ with $p > 2$.

OTHER IMPORTANT ISSUES: OPEN PROBLEMS

– Heat equation with non-smooth coefficients on the principal part. Possibly piecewise constant coefficients.

– To exploit the possibility that the potential $a = a(x, t)$ depends both on x and t and not only on x to improve the optimality result. Note for instance that Meshkov also constructs in $3-d$ a potential $a(x, t)$ for the heat equation for which solutions decay as $t \to \infty$ with velocity $\exp(-ct^2)$.

– Heat equations on graphs and networks.

R. DAGER & E. Z. Wave propagation and control in $1-d$ vibrating multi-structures. Springer Verlag. "Mathématiques et Applications", Paris. 2005

- Fully discrete heat equations.
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