Global existence of solutions for some hyperbolic systems arising from chemotaxis

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16 February 2011, BCAM Working Group Line on PDE
What chemotaxis is?

• Movement of cells driven by chemical signals
• Patlak 1953; Keller-Segel 1970

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(Du \nabla u - \chi(u, \phi) \nabla \phi), \\
\frac{\partial \phi}{\partial t} &= Dc \Delta \phi + f(u, \phi).
\end{align*}
\]

• $u$ is the density of bacteria,
• $\phi$ is the density of the chemoattractant.

Many analytical and numerical results [Horstmann, 03 & 04]:
• existence of global solutions vs. finite time blow-up; analysis of the blow-up profile,
• self-similar solutions, traveling waves...
What chemotaxis is?

Movement of cells driven by chemical signals
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\begin{aligned}
\partial_t u &= \text{div} \left( D_u \nabla u - \chi(u, \phi) \nabla \phi \right), \\
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Two hyperbolic models of chemotaxis

Cattaneo-type model (Hillen)

\[
\begin{align*}
\partial_t u + \text{div}(uV) &= 0, \\
\partial_t(uV) + \gamma^2 \nabla u &= u \nabla \phi - uV, \\
\partial_t \phi &= D \partial_{xx} \phi + f(u, \phi).
\end{align*}
\]
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\end{aligned}
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Euler-type model (Gamba–Preziosi)

\[
\begin{aligned}
\partial_t u + \text{div}(uV) &= 0, \\
\partial_t (uV) + \text{div}(uV \otimes V) + \nabla P(u) &= u \nabla \phi - uV, \\
\partial_t \phi &= D \partial_{xx} \phi + f(u, \phi).
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Biological motivations: tomorrow talk
Two hyperbolic models of chemotaxis

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\end{align*}
\]

Biological motivations: tomorrow talk
The simplest case

\[
\left\{
\begin{array}{l}
\partial_t u + \partial_x v = 0, \\
\partial_t v + \partial_x u = \partial_x \phi u - v, \\
\partial_t \phi - D \partial_{xx} \phi = u - \phi
\end{array}
\right.
\]
The simplest case

\[ \begin{align*}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \partial_x u &= \partial_x \phi u - v, \\
\partial_t \phi - D \partial_{xx} \phi &= u - \phi
\end{align*} \] (1)

go to the blackboard!
A general statement


\[
\begin{aligned}
\partial_t u + \nabla \cdot v &= 0, \\
\partial_t v + \gamma^2 \nabla u &= h(\phi, \nabla \phi)g(u) - (\beta + \bar{b}(\phi, \nabla \phi))v, \\
\partial_t \phi &= \Delta \phi + au - b\phi + \bar{f}(u, \phi).
\end{aligned}
\]

Where \( \bar{b}(\phi, \nabla \phi), h(\phi, \nabla \phi), g(u) \) have linear growth and \( \bar{f}(u, \phi) \) is quadratic
Theorem Under the above assumptions, there exists an $\varepsilon_0 > 0$ such that, if
\[
\| u_0 \|_{H^s}, \| u_0 \|_{L^1}, \| v_0 \|_{H^s}, \| v_0 \|_{L^1}, \| \phi_0 \|_{H^{s+1}}, \| \phi_0 \|_{W^{1,\infty}} \leq \varepsilon_0,
\]
\[\Rightarrow \exists! \text{ global solution to the Cauchy problem (2), for } s \geq \left\lceil \frac{n}{2} \right\rceil + 1:
\]
\[u \in C([0, \infty), H^s(\mathbb{R}^n)), \quad v \in C([0, \infty), H^s(\mathbb{R}^n)), \quad \phi \in C([0, \infty), H^{s+1}(\mathbb{R}^n))\]

Moreover for the solution $(u, v, \phi)$ the following decay rates are satisfied for $k = 0, \ldots, s$
\[
\| u(t) \|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \| u(t) \|_{L^2} \sim t^{-\frac{n}{4}}, \quad \| D_x^k u(t) \|_{L^2} \sim t^{-\delta_k},
\]
\[
\| v(t) \|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \| v(t) \|_{L^2} \sim t^{-\min\left\{\frac{n}{2}, \frac{n}{4} + \frac{1}{2}\right\}}, \quad \| D_x^k v(t) \|_{L^2} \sim t^{-\nu_k},
\]
\[
\| \phi(t) \|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \| D_x^1 \phi(t) \|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \| D_x^{k+1} \phi(t) \|_{L^2} \sim t^{-\delta_k},
\]
\[
\| \phi(t) \|_{L^2} \sim t^{-\frac{n}{4}},
\]
\[
\| D_x^k \phi(t) \|_{L^2} \sim t^{-\delta_k},
\]
Comparison with the diffusive case

Cattaneo System

\begin{align*}
\partial_t u + \nabla \cdot v &= 0, \\
\partial_t v + \nabla u &= -\beta v + h(\phi, \nabla \phi)g(u), \\
\partial_t \phi &= \Delta \phi + au - b\phi + \tilde{f}(u, \phi)
\end{align*}
Comparison with the diffusive case

Cattaneo System

\begin{align*}
\partial_t u + \nabla \cdot \nu &= 0, \\
\partial_t \nu + \nabla u &= -\beta \nu + h(\phi, \nabla \phi) g(u), \\
\partial_t \phi &= \Delta \phi + au - b\phi + \bar{f}(u, \phi)
\end{align*}

Keller-Segel System

\begin{align*}
\beta \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \cdot (h(\tilde{\phi}, \nabla \tilde{\phi}) g(\tilde{u})) &= 0 \\
\partial_t \tilde{\phi} &= \Delta \tilde{\phi} + a\tilde{u} - b\tilde{\phi} + \bar{f}(\tilde{u}, \tilde{\phi})
\end{align*}
Comparison with the diffusive case

Asymptotic convergence Let \((u, v, \phi)\) and \((\tilde{u}, \tilde{\phi})\) be the global solutions respectively to the Cattaneo and Keller-Segel systems. Then there exist \(\varepsilon_0, L > 0\) such that, if

\[
\|u_0\|_{H^s}, \|u_0\|_{L^1}, \|v_0\|_{H^s}, \|v_0\|_{L^1}, \|\phi_0\|_{H^{s+1}}, \|\phi_0\|_{W^{1,\infty}} \leq \varepsilon_0
\]

then, for all \(t > 0\),

\[
\sup_{(0,t)} \left\{ \max\{1, s^{\delta}\} \left( \|u(s) - \tilde{u}(s)\|_{L^2} + \|\phi(s) - \tilde{\phi}(s)\|_{L^2} \right) \right\} \leq L
\]

where \(\delta = \min\{\frac{n}{4} + \frac{1}{2}, \frac{n}{2}\}\).
The Gamba-Preziosi model

\[ \begin{align*}
\partial_t \tilde{\rho} + \partial_x (\tilde{\nu}) &= 0, \\
\partial_t (\tilde{\nu}) + \partial_x \left( \frac{\tilde{\nu}^2}{\tilde{\rho}} + P(\tilde{\rho}) \right) &= \mu \tilde{\rho} \partial_x \tilde{\phi} - \alpha \tilde{\nu}, \\
\partial_t \tilde{\phi} &= D \partial_{xx} \tilde{\phi} + a \tilde{\rho} - \frac{\tilde{\phi}}{\tau},
\end{align*} \]

with \( P'(\rho) > 0 \)
Consider solutions of the form \((\tilde{\rho}, \tilde{v}, \tilde{\phi}) = (\rho + \bar{\rho}, v, \phi + \bar{\phi})\), where \((\bar{\rho}, 0, \bar{\phi})\) is a constant solution

\[
\begin{align*}
\partial_t \rho + \partial_x v &= 0, \\
\partial_t v + \partial_x \left( \frac{v^2}{\rho + \bar{\rho}} + P(\rho + \bar{\rho}) \right) &= \mu(\rho + \bar{\rho}) \partial_x \phi - \alpha v, \\
\partial_t \phi &= D \partial_{xx} \phi + a \rho - \frac{\phi}{\tau}.
\end{align*}
\]
The Gamba-Preziosi model

**Theorem** We consider the Cauchy problem associated to the Gamba-Preziosi system, with small initial data \((\rho_0, v_0) \in H^2(\mathbb{R})\) and \(\phi_0 \in H^2(\mathbb{R})\). If \(\|(\rho_0, v_0)\|_{H^2(\mathbb{R})}, \|\phi_0\|_{H^2(\mathbb{R})}\) and \(\bar{\rho}\) are sufficiently small, then there exists a unique global solution \((\rho, v, \phi)\) to the Gamba-Preziosi system s.t.:

\[
(\rho, v) \in C([0, \infty), H^2(\mathbb{R})), \quad \phi \in C([0, \infty), H^2(\mathbb{R})) \cap L^2([0, \infty), H^3(\mathbb{R}))
\]

and, for each \(T > 0\),

\[
\|(\rho, v)(T)\|_{H^2}^2 + \int_0^T \|\partial_x (\rho, v)(\tau)\|_{H^1}^2 \, d\tau + \int_0^T \|v(\tau)\|_{H^2}^2 \leq C \|(\rho, v)_0\|_{H^2}^2,
\]

\[
\|\phi(T)\|_{H^2}^2 + \int_0^T \|\partial_x \phi(\tau)\|_{H^2}^2 \, d\tau \leq C(\|(\rho, v)_0\|_{H^2}^2 + \|\phi_0\|_{H^2}^2),
\]

where \(C = C(\bar{\rho}, \|(\rho, v)_0\|_{H^2}, \|\phi_0\|_{H^2})\).
The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model
The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model

- The hyperbolic part is quasilinear:
The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model

• The hyperbolic part is quasilinear: to prove the existence no Duhamel formula
The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model

• The hyperbolic part is quasilinear: to prove the existence no Duhamel formula

• Strategy: embed the chemoattract $\phi$ estimates in the Kawashima formalism (Hanouzet-N. proof for the hyperbolic systems)
The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model

- The hyperbolic part is quasilinear: to prove the existence no Duhamel formula
- Strategy: embed the chemoattractant $\phi$ estimates in the Kawashima formalism (Hanouzet-N. proof for the hyperbolic systems)
- Decay estimates can be obtained by the Duhamel formula obtained on the linearized system,
The Gamba-Preziosi model

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- The hyperbolic part is quasilinear: to prove the existence no Duhamel formula
- Strategy: embed the chemoattractant \( \phi \) estimates in the Kawashima formalism (Hanouzet-N. proof for the hyperbolic systems)
- Decay estimates can be obtained by the Duhamel formula obtained on the linearized system, which coincides with the Cattaneo one
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Differences w.r.t. the Cattaneo model

• The hyperbolic part is quasilinear: to prove the existence no Duhamel formula

• Strategy: embed the chemoattract $\phi$ estimates in the Kawashima formalism (Hanouzet-N. proof for the hyperbolic systems)

• Decay estimates can be obtained by the Duhamel formula obtained on the linearized system, which coincides with the Cattaneo one

Blackboard, again!
Decay rates theorem Let \((U, \phi)(t)\) a global (perturbation) solution to the Gamba Preziosi system, with

\[ U_0(x) \in H^{s+1}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \phi_0(x) \in H^{s+1}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \text{for } s \geq 1. \]

Then the following decay estimate holds:

\[ \| U(t) \|_{H^s} + \| \phi(t) \|_{H^{s+1}} \leq C \min\{1, t^{-\frac{1}{4}}\}(E_{s+1} + D_{s+1}) \]

\[ \| U(t) \|_{L^\infty} + \| \phi(t) \|_{L^\infty} \leq C \min\{1, t^{-\frac{1}{2}}\}(E_2 + D_2) \]

where the constant \(C\) depends on the constant state.
The Neumann problem for the Cattaneo model

Let \((U, 0, \Phi)\) be a constant steady state of the Cattaneo model

\((U + u, v, \Phi + \phi)\) perturbed solution

The perturbation \(w = (u, v, \phi)\) satisfies

\[
\begin{cases}
  \partial_t u + \partial_x v = 0, \\
  \partial_t v + \gamma^2 \partial_x u - \chi U \partial_x \phi + \beta v = F_1(\phi, \partial_x \phi) + F_2(\phi, \partial_x \phi) u + F_3(\phi, \partial_x \phi) v, \\
  \partial_t \phi - D \partial_{xx} \phi + b \phi - a u = F_4(u, \phi),
\end{cases}
\]

where

\[
F_1(\phi, \partial_x \phi) := U (g(\Phi + \phi, \partial_x \phi) - \chi \phi) = O\left(\left|\phi, \partial_x \phi\right|^2\right),
\]

\[
F_2(\phi, \partial_x \phi) := g(\Phi + \phi, \partial_x \phi) = O\left(\left|\phi, \partial_x \phi\right|^2\right),
\]

\[
F_3(\phi, \partial_x \phi) := \beta - h(\Phi + \phi, \partial_x \phi) = O\left(\left|\phi, \partial_x \phi\right|^2\right),
\]

\[
F_4(u, \phi) := f(U + u, \Phi + \phi) - a u + b \phi = O\left(\left|u, \phi\right|^2\right)\]
The Neumann problem for the Cattaneo model

Let \((U, 0, \Phi)\) be a constant steady state of the Cattaneo model \((U + u, v, \Phi + \phi)\) perturbed solution

The perturbation \(w = (u, v, \phi)\) satisfies

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\begin{cases}
\partial_t u + \partial_x v = 0, \\
\partial_t v + \gamma^2 \partial_x u - \chi U \partial_x \phi + \beta v \\
= F_1(\phi, \partial_x \phi) + F_2(\phi, \partial_x \phi) u + F_3(\phi, \partial_x \phi) v, \\
\partial_t \phi - D \partial_{xx} \phi + b\phi - a u = F_4(u, \phi),
\end{cases}
\]

where

\[
F_1(\phi, \psi) := U \left( g(\Phi + \phi, \psi) - \chi \psi \right) = O(||(\phi, \psi)||^2),
\]

\[
F_2(\phi, \psi) := g(\Phi + \phi, \psi) = O(||(\phi, \psi)||),
\]

\[
F_3(\phi, \psi) := \beta - h(\Phi + \phi, \psi) = O(||(\phi, \psi)||),
\]

\[
F_4(u, \phi) := f(U + u, \Phi + \phi) - a u + b \phi = O(||(u, \phi)||^2)
\]

\(v = \partial_x \phi = 0, \quad \text{for } x = 0, L\)

\(|(\phi, \psi)| \to 0\)

\(|(u, \phi)| \to 0\)
The linearized problem

Linearizing around a constant state

\[(u, v, \phi) = \left(U, 0, \frac{a}{b} U\right), \quad U \geq 0.\]

\[
\begin{align*}
\partial_t u + \partial_x v &= 0, \\
\partial_t v + \gamma^2 \partial_x u &= \chi U \partial_x \phi - v, \\
\partial_t \phi - D \partial_{xx} \phi &= au - b\phi
\end{align*}
\]

Inserting plane waves like \(u(k) = e^{\lambda(k) t + ikx}\) we find the stability condition to have \(\Re \lambda(k) \leq 0:\).
The linearized problem

Linearizing around a constant state

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\partial_t u + \partial_x v &= 0, \\
\partial_t v + \gamma^2 \partial_x u &= \chi U \partial_x \phi - v, \\
\partial_t \phi - D \partial_{xx} \phi &= au - b\phi
\end{aligned}
\]

Inserting plane waves like \(u(k) = e^{\lambda(k)t + ikx}\) we find the stability condition to have \(\Re \lambda(k) \leq 0:\)

\[
U < \frac{\gamma^2 (b + Dk^2)}{a \chi}
\]
Theorem

Under the previous assumptions, let \((U, 0, \Phi)\) be a constant steady state such that

\[
\chi = \partial_\psi g(\Phi, 0) > 0, \quad \beta = h(\Phi, 0) > 0, \quad \partial_\phi f(U, \Phi) = -b < 0 < a = \partial_u f(U, \Phi)
\]

Assume the stability condition

\[
U < \frac{\gamma^2}{\chi a} \left( b + \frac{D \pi^2}{L^2} \right)
\]
Global existence for the Neumann case

Theorem

Under the previous assumptions, let \((U, 0, \Phi)\) be a constant steady state such that

\[ \chi = \partial_\psi g(\Phi, 0) > 0, \quad \beta = h(\Phi, 0) > 0, \quad \partial_\phi f(U, \Phi) = -b < 0 < a = \partial_u f(U, \Phi) \]

Assume the stability condition

\[ U < \frac{\gamma^2}{\chi a} \left( b + \frac{D \pi^2}{L^2} \right) \]

Let \(w_0 = (u_0, v_0, \phi_0) \in H^1\) the perturbation (with zero mass for \(u_0\)), and \(w\) the corresponding solution. Then there exists \(\varepsilon_0 > 0\) such that, if \(\|w_0\|_{H^1} \leq \varepsilon_0\), then

\[ \|w\|_{H^1(t)} \leq C \|w_0\|_{H^1} e^{-\theta t}. \quad \forall t > 0. \]
A model on a simple network (one node)

\[
\begin{align*}
\partial_t u^i + \partial_x v^i &= 0, \\
\partial_t v^i + \partial_x u^i &= \partial_x \phi^i u^i - v^i, \\
\partial_t \phi^i - D \partial_{xx} \phi^i &= u^i - \phi^i
\end{align*}
\]

• a network of oriented arcs consists of \( M = E \cup U \) intervals (E=enter, U=exit) \([a_i, N], i \in E \) (E=incoming), and \([N, a_i], i \in U \) (U=outcoming), and \( N \) is the node;
• for each \( i = 1, \ldots, M \), \( u_i^\pm(x, t) \) is defined in \([a_i, N] \times [0, T] \in R \times R \);
• initial conditions
• boundary and node conditions (discussed later..)
Diagonal formulation

Let \( u = u^+ + u^- \) and \( v = \lambda(u^+ - u^-) \)

\[
\begin{align*}
\begin{cases}
  u_{i,t}^+ + \lambda_i u_{i,x}^+ &= \frac{1}{2\lambda_i} ((\phi_i^i + \lambda_i)u_i^- - (\lambda_i - \phi_x^i)u_i^+), \\
  u_{i,t}^- - \lambda_i u_{i,x}^- &= -\frac{1}{2\lambda_i} ((\phi_i^i + \lambda_i)u_i^- - (\lambda_i - \phi_x^i)u_i^+).
\end{cases}
\end{align*}
\]

external boundary conditions: no flux conditions

\[
v^i(a_i) = \lambda(u_i^+ - u_i^-) = 0, \phi_x^i(a_i) = 0
\]
Diagonal formulation

Let \( u = u^+ + u^- \) and \( v = \lambda (u^+ - u^-) \)

\[
\begin{align*}
  u_{i,t}^+ + \lambda_i u_{i,x}^+ &= \frac{1}{2\lambda_i}((\phi_x^i + \lambda_i)u_i^- - (\lambda_i - \phi_x^i)u_i^+), \\
  u_{i,t}^- - \lambda_i u_{i,x}^- &= -\frac{1}{2\lambda_i}((\phi_x^i + \lambda_i)u_i^- - (\lambda_i - \phi_x^i)u_i^+).
\end{align*}
\]

external boundary conditions: no flux conditions

\[
v^i(a_i) = \lambda(u_i^+ - u_i^-) = 0, \phi_x^i(a_i) = 0
\]

More general external boundary conditions

\[
u_+^i = \beta_{ai}u_+^i + b_{ai}(t), \quad i \in E
\]

\[
u_-^i = \beta_{ai}u_-^i + b_{ai}(t), \quad i \in U
\]
Flux conservation

\[
\sum_{i \in E} \lambda_i (u_i^+ - u_i^-)(N, t) = \sum_{i \in U} \lambda_i (u_i^+ - u_i^-)(N, t), \quad \sum_i D_i \phi_x^i = 0
\]
Node

Flux conservation

\[
\sum_{i \in E} \lambda_i (u_i^+ - u_i^-)(N, t) = \sum_{i \in U} \lambda_i (u_i^+ - u_i^-)(N, t), \sum_i D_i \phi_x = 0
\]

Conditions in \( N \):

- if \( i \in E \): \( u_i^- (N, t) = \sum_{j \in E} \beta_{i,j} u_j^+(N, t) + \sum_{j \in U} \gamma_{i,j} u_j^-(N, t) \)
- if \( i \in U \): \( u_i^+(N, t) = \sum_{j \in E} \beta_{i,j} u_j^+(N, t) + \sum_{j \in U} \gamma_{i,j} u_j^-(N, t) \)
Node

Flux conservation

\[
\sum_{i \in E} \lambda_i (u_i^+ - u_i^-) (N, t) = \sum_{i \in U} \lambda_i (u_i^+ - u_i^-) (N, t), \quad \sum_i D_i \phi_x^i = 0
\]

Conditions in \( N \):

- if \( i \in E \): \( u_i^- (N, t) = \sum_{j \in E} \beta_{i,j} u_j^+ (N, t) + \sum_{j \in U} \gamma_{i,j} u_j^- (N, t) \)
- if \( i \in U \): \( u_i^+ (N, t) = \sum_{j \in E} \beta_{i,j} u_j^+ (N, t) + \sum_{j \in U} \gamma_{i,j} u_j^- (N, t) \)
- \( D_i \phi_x^i = \sum_j \kappa_{ij} (\phi_j^i (N, t) - \phi_i^i (N, t)) \)
Node

Flux conservation

\[ \sum_{i \in E} \lambda_i (u_i^+ - u_i^-)(N, t) = \sum_{i \in U} \lambda_i (u_i^+ - u_i^-)(N, t), \sum_i D_i \phi_x^i = 0 \]

Conditions in \( N \):

• if \( i \in E \): \( u_i^- (N, t) = \sum_{j \in E} \beta_{i,j} u_j^+ (N, t) + \sum_{j \in U} \gamma_{i,j} u_j^- (N, t) \)
• if \( i \in U \): \( u_i^+ (N, t) = \sum_{j \in E} \beta_{i,j} u_j^+ (N, t) + \sum_{j \in U} \gamma_{i,j} u_j^- (N, t) \)
• \( D_i \phi_x^i = \sum_j \kappa_{ij} (\phi_j^j (N, t) - \phi_i^i (N, t)) \)
• \( \beta_{i,j}, \gamma_{i,j} \in [0, 1]; \kappa_{ij} \geq 0 \)
• \( \sum_{i \in E \cup U} \lambda_i \beta_{i,j} = \lambda_j = \sum_{i \in E \cup U} \lambda_i \gamma_{i,j} \), \( \kappa_{ij} = \kappa_{ji} \)
Analytical results only for the case $\phi_x = \alpha = \text{const.}$

Results obtained in collaboration with Irene Guaraldo (Ph.D Student)
Analytical results only for the case \( \phi_x = \alpha = \text{const.} \).

Results obtained in collaboration with Irene Guaraldo (Ph.D Student)

- If \(|\alpha_i| \leq \lambda\) and \(|\beta_{ai}| \leq 1\), then

\[
\sum_{i=1}^{M} \int_{a_i}^{N} |u_i^+| + |u_i^-| \, dx \leq \sum_{i=1}^{M} \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| \, dx + \sum_{i=1}^{M} \lambda_i \int_{0}^{T} |b_{ai}(t)| \, dt
\]
\[
\int_{a_i}^N |u_i^+| + |u_i^-| \, dx \leq \int_{a_i}^N |u_{i,0}^+| + |u_{i,0}^-| \, dx \\
+ \lambda_i \int_0^T (|u_i^+(a_i)| - |u_i^-(a_i)|) \, dt - \lambda_i \int_0^T (|u_i^+(N)| - |u_i^-(N)|) \, dt \\
\leq \int_{a_i}^N |u_{i,0}^+| + |u_{i,0}^-| \, dx + \lambda_i \int_0^T (|\beta_{a_i} u_i^- + b_{a_i}(t)| - |u_i^-(a_i)|) \, dt \\
- \lambda_i \int_0^T (|u_i^+(N)| - \sum_{j \in E} \beta_{i,j} u_i^+(N, t) + \sum_{j \in U} \gamma_{i,j} u_i^-(N, t)) \, dt \\
\leq \int_{a_i}^N |u_{i,0}^+| + |u_{i,0}^-| \, dx + \lambda_i \int_0^T (|\beta_{a_i} - 1| |u_i^-(a_i)|) \, dt + \int_0^T \lambda_i |b_{a_i}(t)| \, dt \\
- \lambda_i \int_0^T (|u_i^+(N)| - \sum_{j \in E} \beta_{i,j} u_i^+(N, t) + \sum_{j \in U} \gamma_{i,j} u_i^-(N, t)) \, dt \\
\leq \int_{a_i}^N |u_{i,0}^+| + |u_{i,0}^-| \, dx + \int_0^T \lambda_i |b_{a_i}(t)| \, dt \\
- \lambda_i \int_0^T (|u_i^+(N)| - \sum_{j \in E} \beta_{i,j} u_i^+(N, t) + \sum_{j \in U} \gamma_{i,j} u_i^-(N, t)) \, dt; 
\]
Analytical results only for the case $\phi_x = \alpha = \text{const.}$

Results obtained in collaboration with Irene Guaraldo (Ph.D Student)

- If $|\alpha_i| \leq \lambda$ and $|\beta_{a_i}| \leq 1$, then

$$\sum_{i=1}^{M} \int_{a_i}^{N} |u_i^+| + |u_i^-| \, dx \leq \sum_{i=1}^{M} \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| \, dx + \sum_{i=1}^{M} \lambda_i \int_{0}^{T} |b_{a_i}(t)| \, dt$$
Analytical results only for the case $\phi_x = \alpha = \text{const.}$

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- If $|\alpha_i| \leq \lambda$ and $|\beta_{ai}| \leq 1$, then
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  \]

- Uniformly Bounded time derivatives in $L^1$
Analytical results only for the case $\phi_x = \alpha = \text{const.}$

Results obtained in collaboration with Irene Guaraldo (Ph.D Student)

- If $|\alpha_i| \leq \lambda$ and $|\beta_{a_i}| \leq 1$, then
  $$\sum_{i=1}^{M} \int_{a_i}^{N} |u^+_i| + |u^-_i| \, dx \leq \sum_{i=1}^{M} \int_{a_i}^{N} |u^+_{i,0}| + |u^-_{i,0}| \, dx + \sum_{i=1}^{M} \lambda_i \int_{0}^{T} |b_{a_i}(t)| \, dt$$

- Uniformly Bounded time derivatives in $L^1$
- Uniformly Bounded space derivatives in $L^1$
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- If $|\alpha_i| \leq \lambda$ and $|\beta_{a_i}| \leq 1$, then
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  \]

- Uniformly Bounded time derivatives in $L^1$
- Uniformly Bounded space derivatives in $L^1$
- Global existence (and uniqueness) by an approximation procedure (on the Node conditions)
Open problems and perspectives

• 1D Cauchy problem: solutions with big data
• MultiD Cauchy problem: Blow-up in finite time?
• Neumann 1D problem: stability of non constant stationary solutions
• Network problem: general chemoattractant.

Work in progress with I. Guaraldo. Analysis of the semigroup generator for the linear problem
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