Global existence of solutions for some hyperbolic systems arising from chemotaxis

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Patlak 1953; Keller-Segel 1970

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- u is the density of bacteria,
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Many analytical and numerical results [Horstmann, 03 & 04]:

- existence of global solutions vs. finite time blow-up; analysis of the blow-up profile,
- self-similar solutions, traveling waves...

Two hyperbolic models of chemotaxis

Cattaneo-type model (Hillen)

$$\begin{cases} \partial_t u + \mathrm{div}(uV) = 0, \\ \partial_t(uV) + \gamma^2 \nabla u = u\nabla\phi - uV, \\ \partial_t \phi = D\,\partial_{xx}\phi + f(u,\phi). \end{cases}$$

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Biological motivations: tomorrow talk

The simplest case

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = \partial_x \phi \, u - v, \\ \partial_t \phi - D \, \partial_{xx} \phi = u - \phi \end{cases} \tag{1}$$

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go to the blackboard!

A general statement

Guarguaglini, Mascia, Ribot, N. DCDS-B 09: 1D; Di Russo, Ph.D. Thesis dec. 2010): MultiD

$$\begin{cases} \partial_t u + \nabla \cdot v = 0, \\ \\ \partial_t v + \gamma^2 \nabla u = h(\phi, \nabla\phi) g(u) - (\beta + \bar{b}(\phi, \nabla\phi))v, \\ \\ \partial_t \phi = \Delta\phi + au - b\phi + \bar{f}(u, \phi). \end{cases} \tag{2}$$

Where $\bar{b}(\phi, \nabla\phi)$, $h(\phi, \nabla\phi)$, $g(u)$ have linear growth and $\bar{f}(u, \phi)$ is quadratic

Theorem Under the above assumptions, there exists an $\varepsilon_0 > 0$ such that, if

$$\|u_0\|_{H^s}, \|u_0\|_{L^1}, \|v_0\|_{H^s}, \|v_0\|_{L^1}, \|\phi_0\|_{H^{s+1}}, \|\phi_0\|_{W^{1,\infty}} \leq \varepsilon_0,$$

$\Rightarrow \exists!$ global solution to the Cauchy problem (2), for $s \geq \left[\frac{n}{2}\right] + 1$:

$$u \in C([0,\infty), H^s(\mathbb{R}^n)), \; v \in C([0,\infty), H^s(\mathbb{R}^n)), \; \phi \in C([0,\infty), H^{s+1}(\mathbb{R}^n))$$

Moreover for the solution (u, v, ϕ) the following decay rates are satisfied for $k = 0, \ldots, s$

$$\|u(t)\|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \|u(t)\|_{L^2} \sim t^{-\frac{n}{4}}, \quad \|D_x^k u(t)\|_{L^2} \sim t^{-\delta_k},$$

$$\|v(t)\|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \|v(t)\|_{L^2} \sim t^{-\min\left\{\frac{n}{2}, \frac{n}{4}+\frac{1}{2}\right\}}, \quad \|D_x^k v(t)\|_{L^2} \sim t^{-\nu_k},$$

$$\|\phi(t)\|_{L^\infty} \sim t^{-\frac{n}{2}}, \quad \|D_x^1 \phi(t)\|_{L^\infty} \sim t^{-\frac{n}{2}},,$$

$$\|\phi(t)\|_{L^2} \sim t^{-\frac{n}{4}}, \quad \|D_x^{k+1} \phi(t)\|_{L^2} \sim t^{-\delta_k},$$

Comparison with the diffusive case

Cattaneo System

$$\begin{cases} \partial_t u + \nabla \cdot v = 0, \\ \partial_t v + \nabla u = -\beta v + h(\phi, \nabla\phi) g(u), \\ \partial_t \phi = \Delta\phi + au - b\phi + \bar{f}(u, \phi) \end{cases}$$

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Keller-Segel System

$$\begin{cases} \beta \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \cdot (h(\tilde{\phi}, \nabla\tilde{\phi}) g(\tilde{u})) = 0 \\ \partial_t \tilde{\phi} = \Delta\tilde{\phi} + a\tilde{u} - b\tilde{\phi} + \overline{f}(\tilde{u}, \tilde{\phi}), \end{cases}$$

Comparison with the diffusive case

Asymptotic convergence Let (u, v, ϕ) and $(\tilde{u}, \tilde{\phi})$ be the global solutions respectively to the Cattaneo and Keller-Segel systems. Then there exist $\varepsilon_0, L > 0$ such that, if

$$\|u_0\|_{H^s}, \|u_0\|_{L^1}, \|v_0\|_{H^s}, \|v_0\|_{L^1}, \|\phi_0\|_{H^{s+1}}, \|\phi_0\|_{W^{1,\infty}} \leq \varepsilon_0$$

then, for all $t > 0$,

$$\sup_{(0,t)} \left\{ \max\{1, s^\delta\} \left(\|u(s) - \tilde{u}(s)\|_{L^2} + \|\phi(s) - \tilde{\phi}(s)\|_{L^2} \right) \right\} \leq L$$

where $\delta = \min\{\frac{n}{4} + \frac{1}{2}, \frac{n}{2}\}$.

The Gamba-Preziosi model

C. Di Russo - A. Sepe, in preparation 2011 (C. Di Russo Ph.D. Thesis

$$\begin{cases} \partial_t \tilde{\rho} + \partial_x(\tilde{v}) = 0, \\ \\ \partial_t(\tilde{v}) + \partial_x \left(\frac{\tilde{v}^2}{\tilde{\rho}} + P(\tilde{\rho}) \right) = \mu \tilde{\rho} \partial_x \tilde{\phi} - \alpha \tilde{v}, \\ \\ \partial_t \tilde{\phi} = D \partial_{xx} \tilde{\phi} + a\tilde{\rho} - \frac{\tilde{\phi}}{\tau}, \end{cases}$$

with $P'(\rho) > 0$

The Gamba-Preziosi model

Consider solutions of the form $(\tilde{\rho}, \tilde{v}, \tilde{\phi}) = (\rho + \bar{\rho}, v, \phi + \bar{\phi})$, where $(\bar{\rho}, 0, \bar{\phi})$ is a constant solution

$$\begin{cases} \partial_t \rho + \partial_x v = 0, \\ \\ \partial_t v + \partial_x \left(\frac{v^2}{\rho + \bar{\rho}} + P(\rho + \bar{\rho}) \right) = \mu(\rho + \bar{\rho}) \partial_x \phi - \alpha v, \\ \\ \partial_t \phi = D \partial_{xx} \phi + a\rho - \frac{\phi}{\tau}. \end{cases}$$

The Gamba-Preziosi model

Theorem We consider the Cauchy problem associated to the Gamba-Preziosi system, with small initial data $(\rho_0, v_0) \in H^2(\mathbb{R})$ and $\phi_0 \in H^2(\mathbb{R})$. If $\|(\rho_0, v_0)\|_{H^2(\mathbb{R})}$, $\|\phi_0\|_{H^2(\mathbb{R})}$ and $\bar{\rho}$ are sufficiently small, then there exists a unique global solution (ρ, v, ϕ) to the Gamba-Preziosi system s.t.:

$$(\rho, v) \in C([0,\infty), H^2(\mathbb{R})), \quad \phi \in C([0,\infty), H^2(\mathbb{R})) \cap L^2([0,\infty), H^3(\mathbb{R}))$$

and, for each $T > 0$,

$$\|(\rho, v)(T)\|_{H^2}^2 + \int_0^T \|\partial_x(\rho, v)(\tau)\|_{H^1}^2 \, d\tau + \int_0^T \|v(\tau)\|_{H^2}^2 \leq C \|(\rho, v)_0\|_{H^2}^2 , \tag{3}$$

$$\|\phi(T)\|_{H^2}^2 + \int_0^T \|\partial_x \phi(\tau)\|_{H^2}^2 \, d\tau \leq C(\|(\rho, v)_0\|_{H^2}^2 + \|\phi_0\|_{H^2}^2),$$

where $C = C(\bar{\rho}, \|(\rho, v)_0\|_{H^2}, \|\phi_0\|_{H^2})$

The Gamba-Preziosi model

Differences w.r.t. the Cattaneo model

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Blackboard, again!

Decay rates

Decay rates theorem Let $(U,\phi)(t)$ a global (perturbation) solution to the Gamba Preziosi system, with

$$U_0(x) \in H^{s+1}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \phi_0(x) \in H^{s+1}(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ for } s \geq 1.$$

Then the following decay estimate holds:

$$\|U(t)\|_{H^s} + \|\phi(t)\|_{H^{s+1}} \leq C \min\{1, t^{-\frac{1}{4}}\}(E_{s+1} + D_{s+1})$$

$$\|U(t)\|_{L^\infty} + \|\phi(t)\|_{L^\infty} \leq C \min\{1, t^{-\frac{1}{2}}\}(E_2 + D_2)$$

where the constant C depends on the constant state.

The Neumann problem for the Cattaneo model

Let $(U, 0, \Phi)$ be a constant steady state of the Cattaneo model
$(U + u, v, \Phi + \phi)$ perturbed solution
The perturbation $w = (u, v, \phi)$ satisfies

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \gamma^2 \partial_x u - \chi U \partial_x \phi + \beta v \\ \qquad = F_1(\phi, \partial_x \phi) + F_2(\phi, \partial_x \phi)\, u + F_3(\phi, \partial_x \phi)\, v, \\ \partial_t \phi - D \partial_{xx} \phi + b\phi - a u = F_4(u, \phi), \end{cases}$$

$v = \partial_x \phi = 0,$
for $x = 0, L$

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where

$$\begin{aligned} F_1(\phi, \psi) &:= U\left(g(\Phi + \phi, \psi) - \chi\psi\right) = O(|(\phi, \psi)|^2), \\ F_2(\phi, \psi) &:= g(\Phi + \phi, \psi) = O(|(\phi, \psi)|), \qquad\qquad |(\phi, \psi)| \to 0 \\ F_3(\phi, \psi) &:= \beta - h(\Phi + \phi, \psi) = O(|(\phi, \psi)|), \end{aligned}$$

$$F_4(u, \phi) := f(U + u, \Phi + \phi) - a\, u + b\, \phi = O(|(u, \phi)|^2) \qquad\qquad |(u, \phi)| \to 0$$

The linearized problem

Linearizing around a constant state

$$(u, v, \phi) = \left(U, 0, \frac{a}{b}\, U\right), \qquad U \geq 0.$$

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Inserting plane waves like $u(k) = e^{\lambda(k)t+ikx}$ we find the stability condition to have $\Re\lambda(k) \leq 0$:

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Inserting plane waves like $u(k) = e^{\lambda(k)t + ikx}$ we find the stability condition to have $\Re\lambda(k) \leq 0$:

$$\boxed{U < \frac{\gamma^2(b + Dk^2)}{a\,\chi}}$$

Global existence for the Neumann case

Theorem

Under the previous assumptions, let $(U, 0, \Phi)$ be a constant steady state such that

$$\chi = \partial_\psi g(\Phi, 0) > 0,\ \beta = h(\Phi, 0) > 0,\ \partial_\phi f(U, \Phi) = -b < 0 < a = \partial_u f(U, \Phi)$$

Assume the stability condition

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$$U < \frac{\gamma^2}{\chi\, a}\left(b + \frac{D\pi^2}{L^2}\right)$$

Let $w_0 = (u_0, v_0, \phi_0) \in H^1$ the perturbation (with zero mass for u_0), and w the corresponding solution. Then there exists $\varepsilon_0 > 0$ such that, if $\|w_0\|_{H^1} \le \varepsilon_0$, then

$$\|w\|_{H^1}(t) \le C\,\|w_0\|_{H^1}\, e^{-\theta\, t}. \qquad \forall\, t > 0.$$

A model on a simple network (one node)

$$\begin{cases} \partial_t u^i + \partial_x v^i = 0, \\ \partial_t v^i + \partial_x u^i = \partial_x \phi^i \, u^i - v^i, \\ \partial_t \phi^i - D \, \partial_{xx} \phi^i = u^i - \phi^i \end{cases}$$

- a network of oriented arcs consists of $M = E \cup U$ intervals (E=enter, U=exit) $[a_i, N]$, $i \in E$ (E =incoming), and $[N, a_i]$, $i \in U$ (U =outcoming), and N is the node;
- for each $i = 1, \ldots, M$, $u_i^{\pm}(x,t)$ is defined in $[a_i, N] \times [0, T] \in R \times R$
- initial conditions
- boundary and node conditions (discussed later..)

Diagonal formulation

Let $u = u^+ + u^-$ and $v = \lambda(u^+ - u^-)$

$$\begin{cases} u^+_{i,t} + \lambda_i u^+_{i,x} = \dfrac{1}{2\lambda_i}((\phi^i_x + \lambda_i)u^-_i - (\lambda_i - \phi^i_x)u^+_i), \\ u^-_{i,t} - \lambda_i u^-_{i,x} = -\dfrac{1}{2\lambda_i}((\phi^i_x + \lambda_i)u^-_i - (\lambda_i - \phi^i_x)u^+_i). \end{cases}$$

external boundary conditions: no flux conditions

$$v^i(a_i) = \lambda(u^+_i - u^-_i) = 0, \phi^i_x(a_i) = 0$$

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More general external boundary conditions

$$u^i_+ = \beta_{a_i} u^i_- + b_{a_i}(t), \;\; i \in E$$
$$u^i_- = \beta_{a_i} u^i_+ + b_{a_i}(t), \;\; i \in U$$

Node

Flux conservation

$$\sum_{i\in E}\lambda_i(u_i^+ - u_i^-)(N,t) = \sum_{i\in U}\lambda_i(u_i^+ - u_i^-)(N,t), \;\sum_i D_i\phi_x^i = 0$$

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Conditions in N:

- if $i \in E$: $u_i^-(N,t) = \sum_{j\in E}\beta_{i,j}u_j^+(N,t) + \sum_{j\in U}\gamma_{i,j}u_j^-(N,t)$
- if $i \in U$: $u_i^+(N,t) = \sum_{j\in E}\beta_{i,j}u_j^+(N,t) + \sum_{j\in U}\gamma_{i,j}u_j^-(N,t)$

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- if $i \in U$: $u_i^+(N,t) = \sum_{j\in E}\beta_{i,j}u_j^+(N,t) + \sum_{j\in U}\gamma_{i,j}u_j^-(N,t)$
- $D_i\phi_x^i = \sum_j \kappa_{ij}(\phi^j(N,t) - \phi^i(N,t))$

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- $D_i\phi_x^i = \sum_j \kappa_{ij}(\phi^j(N,t) - \phi^i(N,t)$
- $\beta_{i,j},\ \gamma_{i,j} \in [0,1];\ \kappa_{ij} \geq 0$
- $\sum_{i\in E\cup U}\lambda_i\beta_{i,j} = \lambda_j = \sum_{i\in E\cup U}\lambda_i\gamma_{i,j},\quad \kappa_{ij} = \kappa_{ji}$

Analytical results only for the case $\phi_x = \alpha =$const.

Results obtained in collaboration with Irene Guaraldo (Ph.D Student)

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- If $|\alpha_i| \leq \lambda$ and $|\beta_{a_i}| \leq 1$, then

$$\sum_{i=1}^{M} \int_{a_i}^{N} |u_i^+| + |u_i^-| dx \leq \sum_{i=1}^{M} \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| dx + \sum_{i=1}^{M} \lambda_i \int_0^T |b_{a_i}(t)| dt$$

$$
\begin{aligned}
&\int_{a_i}^{N} |u_i^+| + |u_i^-| dx \le \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| dx \\
&+\lambda_i \int_0^T (|u_i^+(a_i)| - |u_i^-(a_i)|) dt - \lambda_i \int_0^T (|u_i^+(N)| - |u_i^-(N)|) dt \\
&\le \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| dx + \lambda_i \int_0^T (|\beta_{a_i} u_-^i + b_{a_i}(t)| - |u_i^-(a_i)|) dt \\
&-\lambda_i \int_0^T (|u_i^+(N)| - |\sum_{j\in E} \beta_{i,j} u_+^j(N,t) + \sum_{j\in U} \gamma_{i,j} u_-^j(N,t)|) dt \\
&\le \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| dx + \lambda_i \int_0^T (|\beta_{a_i}| - 1)|u_-^i(a_i)| dt + \int_0^T \lambda_i |b_{a_i}(t)| dt \\
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$$\sum_{i=1}^{M} \int_{a_i}^{N} |u_i^+| + |u_i^-| dx \leq \sum_{i=1}^{M} \int_{a_i}^{N} |u_{i,0}^+| + |u_{i,0}^-| dx + \sum_{i=1}^{M} \lambda_i \int_0^T |b_{a_i}(t)| dt$$

- Uniformly Bounded time derivatives in L^1

Analytical results only for the case $\phi_x = \alpha =$const.

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- Uniformly Bounded space derivatives in L^1
- Global existence (and uniqueness) by an approximation procedure (on the Node conditions)

Open problems and perspectives

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- MultiD Cauchy problem: Blow-up in finite time?

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  Work in progress with I. Guaraldo. Analysis of the semigroup generator for the linear problem