Hardy inequalities and applications

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RT1-PDEs Group Activity

Classical Hardy inequality

$$\text{G.H Hardy}: \int_0^\infty |u'|^2 dr \geq \frac{1}{4}\int_0^\infty \frac{u^2}{r^2} dr, \quad \forall u \in H_0^1(0,\infty). \tag{1}$$

In other words,

$$-\partial_{rr}^2 - \frac{1}{4r^2} \geq 0.$$

Hardy, G.H., *An inequality between integrals*, Messenger of Math., 54, 150–156, 1925.

Hardy inequalities

The Hardy inequalities give information on the positivity of the Scrödinger operators like e.g.

$$-\Delta - \frac{\mu}{|x|^2}, \quad -\Delta - \sum_{i\in I} \frac{\mu_i}{|x-x_i|^2}, \quad -\Delta - \frac{\mu}{d^2(x).}$$

For a given domain Ω, $d(x) =$ distance to the boundary $= d(x, \partial\Omega)$.

Important: These operators are interesting when becoming singular.

Regarding this, we discuss and compare three such operators that may appear in the literature:

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Given an open domain $\Omega \subset \mathbb{R}^N$ we distinguish:

1. The case $0 \in \Omega$,

$$L_\mu^1 := -\Delta - \frac{\mu}{|x|^2}$$

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2.

$$L_\mu^2 := -\Delta - \frac{\mu}{d^2(x)}$$

3. The case $0 \in \partial\Omega$,

$$L_\mu^3 := -\Delta - \frac{\mu}{|x|^2}$$

Question-Objective: In order to have

$$L_\mu^i \geq 0, \quad i \in \{1, 2, 3\}$$

what can we say about μ? What is the optimal value?

Definition:

$$L_\mu^i \geq 0 \Leftrightarrow (L_\mu^i u, u)_{L^2(\Omega)} \geq 0, \quad \forall u \in H_0^1(\Omega).$$

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Hardy inequalities

Define optimal values:

$$\mu^i(\Omega) = \inf_{u \neq 0, u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2/|x|^2 dx}, \quad i \in \{1,3\}$$

and

$$\mu^2(\Omega) = \inf_{u \neq 0, u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2/d^2(x) dx}.$$

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Singularity located in interior

Let $N \geq 3$. We consider $\Omega \subset \mathbb{R}^N$ an open domain, with $0 \in \Omega$. Then

$$\mu^1(\Omega) = \Big(\frac{N-2}{2}\Big)^2, \quad \forall N \geq 3,$$

and more precisely:

Lemma (Hardy inequality)

$$\int_\Omega |\nabla \mathbf{u}|^2 \mathbf{dx} \geq \Big(\frac{\mathbf{N-2}}{\mathbf{2}}\Big)^2 \int_\Omega \frac{|\mathbf{u}|^2}{|\mathbf{x}|^2} \mathbf{dx} \quad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega). \tag{2}$$

The constant $(N-2)^2/4$ *is optimal and not attained* ($\inf \neq \min$).

Hardy, G. H. and Littlewood, J. E. and Pólya, G. *Inequalities*, 1952

Remark: $\mu^1(\Omega)$ does not depend on Ω.

For $N = 2$, it arises in a natural way, the inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2 (\log 1/|x|)^2} dx$$

Singularity located in interior

[Proof of Lemma 1] For $N \neq 2$, integrating by parts and using the Hölder inequality we can obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx &= \frac{1}{N-2} \int\limits_{\mathbb{R}^N} u^2 \mathrm{div}\Big(\frac{x}{|x|^2}\Big) dx \\
&= -\frac{1}{N-2} \int\limits_{\mathbb{R}^N} 2u\nabla u \cdot \frac{x}{|x|^2} dx \\
&= -\frac{2}{N-2} \int\limits_{\mathbb{R}^N} \frac{u}{|x|} \Big(\nabla u \cdot \frac{x}{|x|}\Big) dx \\
&\leq \frac{2}{N-2} \Big(\int\limits_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx\Big)^{\frac{1}{2}} \Big(\int\limits_{\mathbb{R}^N} |\nabla u|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Singularity located in interior

Why $\mu^1(\Omega) = \mu^1(\mathbb{R}^N)$?

Steps:

1. 1. $\mu^1(B_R) = \mu(B_1)$ (re-scalling)
2. 2.If Ω is bounded $\Rightarrow$ ex. $R_1 > R_2 > 0$ such that $\mu^1(B_{R_1}) \leq \mu^1(\Omega) \leq \mu^1(B_{R_2})$.
3. 3. $\mu^1(B_1) = \mu^1(\mathbb{R}^N)$:

Consider an approximation U_ε for $\mathbb{R}^N$, and choose the particular subsequence $U_{\frac{1}{n}}$, with $n \in \mathbb{N}, n > 0$. Let $\varphi_n \in C_c^\infty(B_n)$ be a smooth cut-off function such that $0 < \varphi_n \leq 1$,

$$\varphi_n(x) = \begin{cases} 1 & , x \in B_{n-1} \\ 0 & , x \in \mathbb{R}^N \setminus B_n \end{cases} \tag{3}$$

and moreover $|\nabla \varphi_n| \leq m$. We consider the test functions $v_n(x) = U_{\frac{1}{n}} \varphi_n(x)$. Then it follows that $u_n := v_n\left(\frac{\cdot}{n}\right)$ is an approximation for a ball.

Singularity located in interior

What about the difference

$$\int_\Omega |\nabla u|^2 dx - \mu^1(\Omega) \int_\Omega \frac{u^2}{|x|^2} dx?$$

Answer: It is positive..but...
Once Ω is bounded we still have

$$\text{difference } \geq C_1(\Omega) \int_\Omega u^2 dx, \quad (\text{ Brezis-Vazquez 1997}),$$

$$\text{difference } \geq C_2(\Omega) ||u||^2_{H^s(\Omega)} dx, \quad s \in (0,1), \quad (\text{ Vazquez-Zuazua 2000}),$$

$$\text{difference } \geq C_3(\Omega) \int_\Omega \frac{u^2}{|x|^2 (\log(R/|x|))^2} dx, \quad \Omega \subset B_R(0), (\text{ Adimurthi-Ramaswamy '02}$$

HI with distance to the boundary

1. If Ω is convex, then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2(x)} dx, \quad \forall u \in H_0^1(\Omega), \quad \text{(Davies)}.$$

   Moreover,

$$\mu^2(\Omega) = \frac{1}{4}.$$

2. If Ω is smooth then

$$\int_\Omega |\nabla u|^2 dx \geq C(\Omega) \int_\Omega \frac{u^2}{d^2(x)} dx, \quad \forall u \in H_0^1(\Omega), \quad \text{(Ancona)}.$$

3. If Ω is smooth then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{d^2(x)} dx \pm C \int_\Omega u^2 dx, \quad \forall u \in H_0^1(\Omega), \quad \text{(Brezis-Marcus)}.$$

Singularity located on to the boundary

Firstly, one can notice that

$$\mu^3(\Omega) \geq \left(\frac{N-2}{2}\right)^2.$$

Question-Objective: Are we able to obtain something more ($>$ instead of $\geq$) ?

$2-d$ cones

Let be $\gamma \in (0,\pi)$.

$$2-d \text{ cone } \mathcal{C}_\gamma : \begin{cases} x_1 = r\cos\theta, \\ x_2 = r\sin\theta, \end{cases} \tag{4}$$

with $r>0$ and $\theta \in (0,\gamma)$.

Theorem (Hardy in cones)

$$\int_{\mathcal{C}_\gamma} |\nabla v|^2 dx \geq \frac{\pi^2}{\gamma^2} \int_{\mathcal{C}_\gamma} \frac{|v|^2}{|x|^2} dx \quad \forall v \in H_0^1(\Omega). \tag{5}$$

Moreover, the constant π^2/γ^2 is optimal in (5).

Optimality is given by the approximating sequence

$$v_\varepsilon(x) = v_\varepsilon(r\cos\theta, r\sin\theta) = u_\varepsilon(r,\theta) := \begin{cases} r^\varepsilon \sin(\frac{\pi}{\gamma}\theta), & |x| \leq 1 \\ r^{-\varepsilon}\sin(\frac{\pi}{\gamma}\theta), & |x| \geq 1. \end{cases} \tag{6}$$

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$N-d$ cones, $N \geq 3$

Let us consider $\pi > \gamma > 0$. By $\mathcal{C}_\gamma$ we define the $N-d$ cone with the aperture 2γ (e.g. [7], pp. 293):

$$\begin{cases} x_1 = r\sin\theta_1 \sin\theta_2 \dots \sin\theta_{N-2}\cos_{N-1} \\ x_2 = r\sin\theta_1 \sin\theta_2 \dots \sin\theta_{N-2}\sin_{N-1} \\ \vdots \\ x_{N-1} = r\sin\theta_1\sin\theta_2 \\ x_N = r\cos\theta_1 \end{cases} \tag{7}$$

with $0 \leq \theta_i \leq 2\pi$ for $2 \leq i \leq N-2$, $0 < \theta_1 \leq \gamma$, $0 \leq \theta_{N-1} \leq 2\pi$, $r > 0$.

Theorem ($N-d$ convex cones)

If $0 < \gamma \leq \frac{\pi}{2}$ *then*

$$\forall u \in C_c^\infty(\mathcal{C}_\gamma), \quad \int_{\mathcal{C}_\gamma} |\nabla u|^2 dx \geq \left(\frac{(N-2)^2}{4} + \frac{(N-1)\pi^2}{4\gamma^2}\right)\int_{\mathcal{C}_\gamma} \frac{|u|^2}{|x|^2}dx. \tag{8}$$

$N-d$ cones

In particular, for $\gamma = \pi/2$ we recover the optimal inequality in the half-space (see Filippas&Tertikas[2]).

Theorem (Filippas&Tertikas)

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{|x|^2} dx, \quad \forall u \in H^1_0(\mathbb{R}^N_+). \tag{9}$$

The constant $N^2/4$ in inequality (9) is sharp due to

$$u_\varepsilon(x) = \begin{cases} x_N & |x| \leq 1, x \in \mathbb{R}^N_+ \\ |x|^{\alpha_\varepsilon} x_N & |x| \geq 1, x \in \mathbb{R}^N_+ \end{cases} \tag{10}$$

where $\alpha_\varepsilon := -N/2 - \varepsilon$, $\varepsilon > 0$.

what happens in concave cones??

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Theorem

Let $\varepsilon > 0$. *There exists* $\pi/2+\varepsilon > \gamma = \gamma(\varepsilon) > \pi/2$ *such that* $\forall u \in C_c^\infty(\mathcal{C}_\gamma)$,

$$\int_{\mathcal{C}_\gamma} |\nabla u|^2 dx \geq \left(\frac{(N-2)^2}{4} + \frac{(N-1)\pi^2}{4\gamma^2} - o(1)\right) \int_{\mathcal{C}_\gamma} \frac{|u|^2}{|x|^2} dx. \tag{11}$$

Singularity on the boundary

If Ω is a ball then

$$\mu^3(\Omega) = \mu^3(\mathbb{R}^N_+) = \frac{N^2}{4}.$$

If Ω is a domain with tangential hyperplane at the origin and interior ball property at the origin, then

$$\mu^3(\Omega) = \mu^3(\mathbb{R}^N_+) = \frac{N^2}{4}.$$

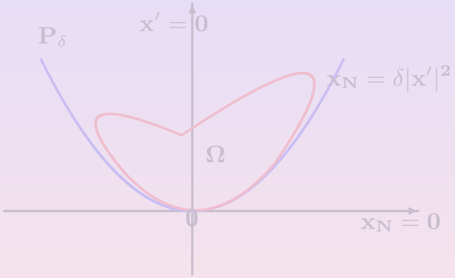
If Ω is an exterior of a bounded domain then

$$\mu^3(\Omega) = \left(\frac{N-2}{2}\right)^2. \quad \text{(Ghoussoub)}.$$

Elliptic geometry at the origin ($\delta > 0$)

Theorem

$$\int_\Omega |\nabla v|^2 dx - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx \geq C(\Omega,\delta)\int_\Omega \frac{v^2}{|x|}dx \quad \forall v \in H_0^1(\Omega). \tag{12}$$



sketch of the proof: Rewriting the l.h.s in terms of the new variable $v(x) = (x_N - \delta|x'|^2)u(x)|x|^{-N/2}$.

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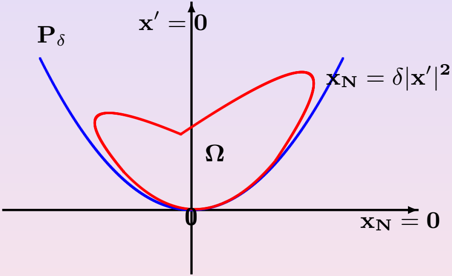

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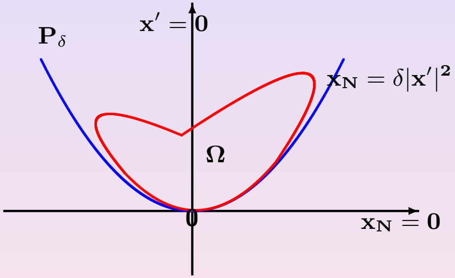

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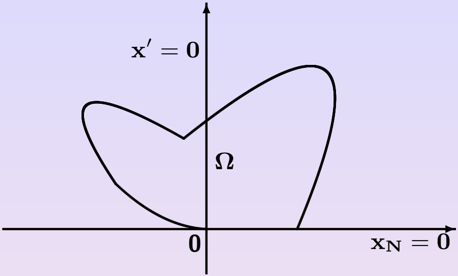
cylindrical geometry at the origin ($\delta = 0$)



Theorem

If L is a positive number such that $L > \sup_{x\in\overline{\Omega}}|x|$, then

$$\int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx + \frac{1}{4}\int_\Omega \frac{|v|^2}{(|x|^2\log(L/|x|))^2}dx \quad \forall v \in H_0^1(\Omega). \tag{13}$$

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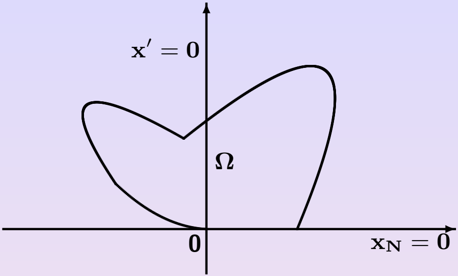

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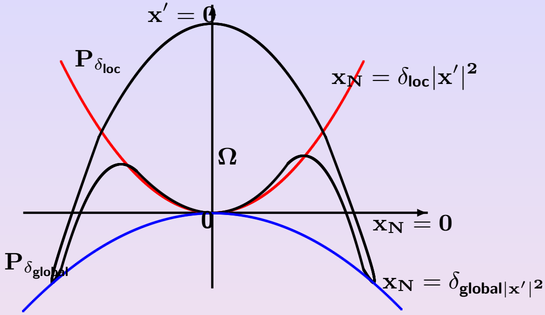
with local elliptic geometry at the origin

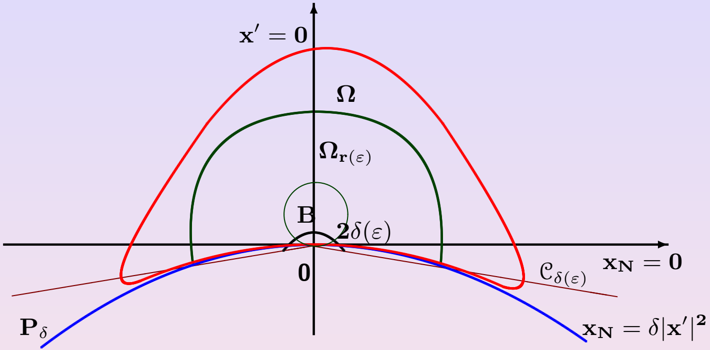


Theorem

There exists a positive constant $C(\Omega)$ such that

$$C(\Omega)\int_\Omega v^2dx + \int_\Omega |\nabla v|^2dx \geq \frac{N^2}{4}\int_\Omega \frac{|v|^2}{|x|^2}dx \quad \forall v \in H_0^1. \tag{14}$$

with hyperbolic geometry at the origin



with hyperbolic geometry at the origin

Theorem

For any $\varepsilon > 0$ there exists a positive constant $C(\Omega, \varepsilon)$ such that the following inequality holds:

$$\boxed{C(\Omega,\varepsilon)\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx \geq \left(\frac{N^2}{4} - \varepsilon\right)\int_\Omega \frac{v^2}{|x|^2}, \quad \forall v \in H_0^1(\Omega).} \tag{15}$$

and the constant $C(\Omega, \varepsilon)$ goes to infinity as ε goes to zero.

ingredients for the proof: local approximations by concave cones near by the origin, decomposition (cut-off)

with hyperbolic geometry at the origin

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Applications

1. classical subject with applications to PDE´s
2. $\frac{1}{|x|^2} \quad \rightarrow \quad \text{inverse-squared singular potential;}$ appears in the asymptotic behaviour of branches of solutions in bifurcation problems (see Brezis&Vazquez-[1], Tertikas-[8]), molecular physics (Leblond-[5]), quantum cosmology (Esteban& Berestychi-[3]), linearization of combustion models (Azorero&Peral-[6]).
3. physics (relativity theory, quantum mechanics).

1. Asymptotic analysis of the Hardy constant in (11): $o(1) = ?$
2. Determine sharp inequalities in concave cones?!
3. Study the controllability of evolution problems with singular quadratic potentials localized on the boundary
4. Why does the singularity produces a discontinues process in the evolution of the Hardy constant, when passing from the interior to the boundary ?

Thanks!

- H. Brezis and L. Vázquez, J., *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.
- S. Filippas, A. Tertikas, and J. Tidblom, *On the structure of Hardy-Sobolev-Maz'ya inequalities*, 2009.
- M.J. Esteban H. Berestycki, *Existence and bifurcation of solutions for an elliptic degenerate problem*, J. Differential Equations **134**, no. 1.
- J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 8, Masson, Paris, 1988.
- J.M Lévy-Leblond, *Electron capture by polar molecules*, Phys. Rev. **153**, no. 1.
- García Azorero J. P. and Peral Alonso I., *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equations **144** (1998), no. 2, 441–476.
- Elias M. Stein and Rami Shakarchi, *Fourier analysis*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003, An introduction.
- A. Tertikas, *Critical phenomena in linear elliptic problems*, J. Funct. Anal. **154** (1998), no. 1, 42–66.
- J. Vancostenoble and E. Zuazua, *Hardy inequalities, observability and control for the wave equation and Schrödinger equations with singular potentials*, preprint.