An Asymptotic Preserving Scheme for Kolmogorov Equation

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Kolmogorov Equation

$$\partial_t f - \partial_v^2 f - v\partial_x f = 0 \qquad (t, v, x) \in \mathbb{R}_+^* \times \mathbb{R}^2 \tag{1}$$

With a suitable change of variables we can transform the Kolmogorov equation, (1), into the following

$$\partial_s = \partial_{vv} g + \frac{3}{2}\, x\partial_x g + A^2(s)\partial_{xx} g + \frac{1}{2}\, v\, \partial_v g + 2\, A(s)\, \partial_{vx} g,$$

where $A(s) = 1 - e^{-s}$.Then as $A(s) \to 1$ as $s \to \infty$ and we get the following simplified equation

$$\partial_s g = \partial_{vv} g + \frac{3}{2}\, x\partial_x g + \partial_{xx} g + \frac{1}{2}\, v\, \partial_v g + 2\, \partial_{vx} g. \tag{2}$$

Take the initial condition to be

$$g_0(v, x) = 10\, e^{-5\times 10^5 (v^2 + x^2)}. \tag{3}$$

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Generalized Hermite Polynomials

We define the general Hermite polynomial, $H_{\alpha,n}(x)$ for $x \in \mathbb{R}$, as the solution to the following differential equation

$$\partial_x \left(e^{-x^2/\alpha^2} \partial_x H_{\alpha,n}(x) \right) + \lambda_n e^{-x^2/\alpha^2} H_{\alpha,n}(x) = 0, \tag{4}$$

where

$$\lambda_n = \frac{2\,n}{\alpha^2} \tag{5}$$

is the nth eigenvalue associated with $H_{\alpha,n}(x)$.

Properties of generalized Hermite Polynomials

- Shift Operator

$$H_{\alpha,n}(x) = \frac{2n}{\alpha^2} H_{\alpha,n-1}(x).$$

- Recursion

$$H_{\alpha,n}(x) = \frac{2\,x}{\alpha^2} H_{\alpha,n-1}(x) - H'_{\alpha,n-1}(x).$$

- Orthogonality

$$\int_{\mathbb{R}} H_{\alpha,n}(x) H_{a,m}(x) e^{-x^2/\alpha^2}\, dx = 2^{1-3\,n} a^{1-2\,n} n! \sqrt{\pi} \delta_{nm}.$$

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Are Hermite Polynomials Suited to Our Problem?

- Recall:

$$\partial_x \left(e^{-x^2/\alpha^2} \partial_x H_{\alpha,n}(x) \right) + \lambda_n e^{-x^2/\alpha^2} H_{\alpha,n}(x) = 0.$$

- Rewritten:

$$\partial_{xx} H_{\alpha,n}(x) - \frac{2\,x}{\alpha^2} \partial_x H_{\alpha,n} = -\lambda_n H_{\alpha,n}(x).$$

- Hermite polynomials tend to increase instead of decaying as $x \to \infty$
- But we need something more like:

$$\partial_{xx} H_{\alpha,n}(x) + \frac{2\,x}{\alpha^2} \partial_x H_{\alpha,n} = -\lambda_n H_{\alpha,n}(x).$$

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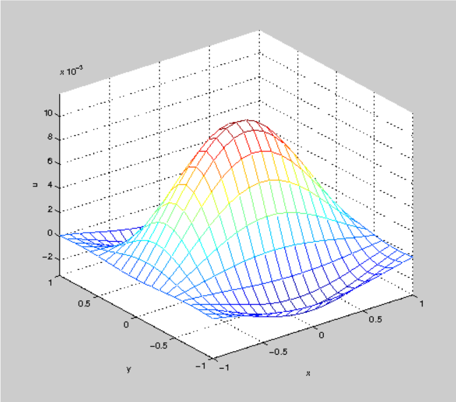
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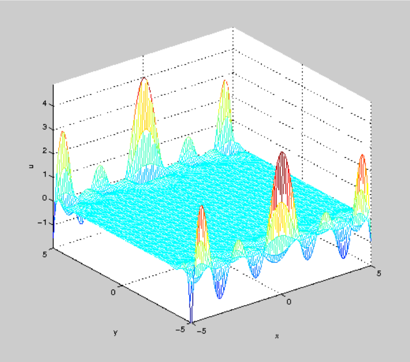
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- This can be rewritten in the following way:

$$e^{-x^2/\alpha^2} \partial_x \left(e^{x^2/\alpha^2} \partial_x u \right) = -\lambda u. \tag{6}$$

- Luckily this ends up just being

$$h_{\alpha,n}(x) = e^{-x^2/\alpha^2} H_{\alpha,n-1}(x). \tag{7}$$

- With these new Hermite polynomials we see that they tend to zero as $x \to \infty$.

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Applying the Method

- Recall the simplified self-silmilar Kolmogorov equation, (2),

$$\partial_s g = \partial_{vv} g + \frac{3}{2}\, x\partial_x g + \partial_{xx} g + \frac{1}{2}\, v\, \partial_v g + 2\, \partial_{vx} g.$$

- Taking $\alpha = 2$ in the v-direction and $\alpha = \frac{2}{\sqrt{3}}$ in the x-direction and rewriting (2):

$$\partial_s g = e^{-v^2/4}\partial_v \left(e^{v^2/4}\partial_v g\right) + e^{-3x^2/4}\partial_x \left(e^{3x^2/4}\partial_x g\right) + 2\, \partial_{vx} g. \qquad (8)$$

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- Take

$$g = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{n,m}(t) h_{2,n}(v) h_{2/\sqrt{3},m}(x),$$

where $h_{\alpha,n}(x)$ indicates the normalized Hermite polynomial.

- Finally applying the Hermite polynomial basis to (8), we have

$$\partial_s a_{n,m}(s) = \lambda_n a_{n,m}(s) + \tau_m a_{n,m}(s) + 2\,\partial_{vx} \sum_{l=0}^{M} \sum_{k=0}^{N} a_{k,l}(s) \int_{\mathbb{R}^2} h_{2,k}(v) h_{2,n}(v) h_{2/\sqrt{3},l}(x) h_{2/\sqrt{3},m}(x) \tag{9}$$

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$$\begin{aligned} \partial_s a_{n,m}(s) = \lambda_n a_{n,m}(s) + \tau_m a_{n,m}(s) \\ + 2\,\partial_{vx} \sum_{l=0}^{M} \sum_{k=0}^{N} a_{k,l}(s) \\ \int_{\mathbb{R}^2} h_{2,k}(v) h_{2,n}(v) h_{2/\sqrt{3},l}(x) h_{2/\sqrt{3},m}(x) \end{aligned} \tag{9}$$