An Asymptotic Preserving Scheme for Kolmogorov Equation

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Kolmogorov Equation

\[ \partial_t f - \partial_v^2 f - v\partial_x f = 0 \quad (t, v, x) \in \mathbb{R}^*_+ \times \mathbb{R}^2 \]  \hspace{1cm} (1)

With a suitable change of variables we can transform the Kolmogorov equation, (1), into the following

\[ \partial_s = \partial_{vv} g + \frac{3}{2} x \partial_x g + A^2(s) \partial_{xx} g + \frac{1}{2} v \partial_v g + 2 A(s) \partial_{vx} g, \]

where \( A(s) = 1 - e^{-s} \). Then as \( A(s) \to 1 \) as \( s \to \infty \) and we get the following simplified equation

\[ \partial_s g = \partial_{vv} g + \frac{3}{2} x \partial_x g + \partial_{xx} g + \frac{1}{2} v \partial_v g + 2 \partial_{vx} g. \]  \hspace{1cm} (2)

Take the initial condition to be

\[ g_0(v, x) = 10 e^{-5 \times 10^5(v^2 + x^2)}. \]  \hspace{1cm} (3)
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Generalized Hermite Polynomials

We define the general Hermite polynomial, \( H_{\alpha,n}(x) \) for \( x \in \mathbb{R} \), as the solution to the following differential equation

\[
\partial_x \left( e^{-x^2/\alpha^2} \partial_x H_{\alpha,n}(x) \right) + \lambda_n e^{-x^2/\alpha^2} H_{\alpha,n}(x) = 0,
\]

where

\[
\lambda_n = \frac{2n}{\alpha^2}
\]

is the \( n^{th} \) eigenvalue associated with \( H_{\alpha,n}(x) \).
Properties of generalized Hermite Polynomials

- **Shift Operator**
  \[ H_{\alpha,n}(x) = \frac{2n}{\alpha^2} H_{\alpha,n-1}(x). \]

- **Recursion**
  \[ H_{\alpha,n}(x) = \frac{2x}{\alpha^2} H_{\alpha,n-1}(x) - H'_{\alpha,n-1}(x). \]

- **Orthogonality**
  \[ \int_{\mathbb{R}} H_{\alpha,n}(x) H_{\alpha,m}(x) e^{-x^2/\alpha^2} \, dx = 2^{1-3n} \alpha^{1-2n} n! \sqrt{\pi} \delta_{nm}. \]
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Are Hermite Polynomials Suited to Our Problem?

- Recall:

\[
\partial_x \left( e^{-x^2/\alpha^2} \partial_x H_{\alpha,n}(x) \right) + \lambda_n e^{-x^2/\alpha^2} H_{\alpha,n}(x) = 0.
\]

- Rewritten:

\[
\partial_{xx} H_{\alpha,n}(x) - \frac{2x}{\alpha^2} \partial_x H_{\alpha,n} = -\lambda_n H_{\alpha,n}(x).
\]

- Hermite polynomials tend to increase instead of decaying as \( x \to \infty \)
- But we need something more like:

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This can be rewritten in the following way:

\[
e^{-x^2/\alpha^2} \partial_x \left( e^{x^2/\alpha^2} \partial_x u \right) = -\lambda u.
\] (6)

Luckily this ends up just being

\[
h_{\alpha,n}(x) = e^{-x^2/\alpha^2} H_{\alpha,n-1}(x).
\] (7)

With these new Hermite polynomials we see that they tend to zero as \(x \to \infty\).
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Applying the Method

- Recall the simplified self-similar Kolmogorov equation, (2),

$$\partial_s g = \partial_{vv} g + \frac{3}{2} x \partial_x g + \partial_{xx} g + \frac{1}{2} v \partial_v g + 2 \partial_{vx} g.$$ 

- Taking $\alpha = 2$ in the $v$-direction and $\alpha = \frac{2}{\sqrt{3}}$ in the $x$-direction and rewriting (2):

$$\partial_s g = e^{-v^2/4} \partial_v \left(e^{v^2/4} \partial_v g \right) + e^{-3x^2/4} \partial_x \left(e^{3x^2/4} \partial_x g \right) + 2 \partial_{vx} g. \quad (8)$$
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Take

\[ g = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{n,m}(t) h_{2,n}(v) h_{2/\sqrt{3},m}(x), \]

where \( h_{\alpha,n}(x) \) indicates the normalized Hermite polynomial.

Finally, applying the Hermite polynomial basis to (8), we have

\[ \partial_s a_{n,m}(s) = \lambda_n a_{n,m}(s) + \tau_m a_{n,m}(s) \]

\[ + 2 \partial_{vx} \sum_{l=0}^{M} \sum_{k=0}^{N} a_{k,l}(s) \]

\[ \int_{\mathbb{R}^2} h_{2,k}(v) h_{2,n}(v) h_{2/\sqrt{3},l}(x) h_{2/\sqrt{3},m}(x) \]
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\[ \int_{\mathbb{R}^2} h_{2,k}(v) h_{2,n}(v) h_{2/\sqrt{3},l}(x) h_{2/\sqrt{3},m}(x) \]

\[ (9) \]