Asymptotic limits for the 1D nonlinear Mindlin-Timoshenko system

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Contents

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Linear Mindlin-Timoshenko system

$$\left| \begin{array}{lll} \dfrac{\rho h^3}{12}\phi'' - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} & Q, \\ \rho h \psi'' - k(\phi + \psi_x)_x = 0 & \text{in} & Q, \\ \textbf{Boundary Conditions} & & \\ \phi(x,0) = \phi_0(x), \quad \phi'(x,0) = \phi_1(x) & \text{in} & (0,L), \\ \psi(x,0) = \psi_0(x), \quad \psi'(x,0) = \psi_1(x) & \text{in} & (0,L). \end{array} \right. \qquad \text{(M-T)}$$

- $Q = (0,L) \times (0,T)$
- ϕ - angle of rotation
- ψ - vertical displacement
- ρ - density, h - thickness of the beam
- $k > 0$ - modulus of elasticity in shear

[1]

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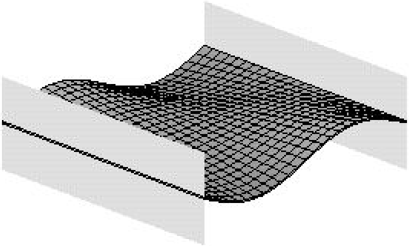
$$[\phi(0,t) = \phi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0 \quad \text{on} \quad (0,T)]$$



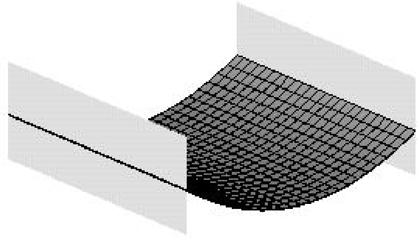
The energy of the Mindlin-Timoshenko system

$$E_k(t) = \frac{1}{2}\int_0^L \left\{ \frac{\rho h^3}{12} \left|\phi'(x,t)\right|^2 + \rho h \left|\psi'(x,t)\right|^2 + \left|\phi_x(x,t)\right|^2 + \right.$$
$$\left. + k \left|\phi(x,t) + \psi_x(x,t)\right|^2 \right\} dx$$

is conservative, that is,

$$E_k(t) = E_k(0).$$

When assuming that the linear filament of the beam remains perpendicular to the deformed middle surface, the transverse shear effects are neglected.



Linear Kirchhoff system

$$\left| \begin{array}{lll} \rho h \psi'' - \dfrac{\rho h^3}{12}\psi''_{xx} + \psi_{xxxx} = 0 & \text{in} & Q, \\ \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{on} & (0,T), \\ \psi_{xxx}(0,\cdot) = \psi_{xxx}(L,\cdot) = 0 & \text{on} & (0,T), \\ \psi(\cdot,0) = \psi_0, \quad \psi'(\cdot,0) = \psi_1 & \text{in} & (0,L). \end{array} \right. \qquad (K)$$

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Notice that the energy of the Kirchhoff system

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \left\{ \rho h \left|\psi'(x,t)\right|^2 + \frac{\rho h^3}{12}\left|\psi'_x(x,t)\right|^2 + \left|\psi_{xx}(x,t)\right|^2 \right\} dx.$$

is also conservative, that is,

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Control problem

$$\left| \begin{array}{lll} \frac{\rho h^3}{12} u'' - u_{xx} + k\left(u + v_x\right) = 0 & \text{in} & Q, \\ \rho h v'' - k\left(u + v_x\right)_x = 0 & \text{in} & Q, \\ u\left(0, \cdot\right) = 0, \quad u\left(L, \cdot\right) = 0 & \text{on} & \left(0, T\right), \\ v_x\left(0, \cdot\right) = \Theta_k, \quad v_x\left(L, \cdot\right) = 0 & \text{on} & \left(0, T\right), \\ u\left(\cdot, 0\right) = u_0, \quad u'\left(\cdot, 0\right) = u_1 & \text{in} & \left(0, L\right), \\ v\left(\cdot, 0\right) = v_0, \quad v'\left(\cdot, 0\right) = v_1 & \text{in} & \left(0, L\right). \end{array} \right.$$

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Problem: given $T > 0$, large enough, and initial data, to find a control Θ_k such that the solution of system satisfies the conditions

$$u\left(\cdot, T\right) = u'\left(\cdot, T\right) = v\left(\cdot, T\right) = v'\left(\cdot, T\right) = 0 \quad \text{in} \quad \left(0, L\right).$$

As $k \to \infty$, the Mindlin-Timoshenko system tends to Kirchhoff system

$$\left| \begin{array}{lll} \rho h v'' - \frac{\rho h^3}{12} v''_{xx} + v_{xxxx} = 0 & \text{in} & Q, \\ v_x(0,\cdot) = 0, \quad v_x(L,\cdot) = 0 & \text{on} & (0,T), \\ v_{xxx}(0,\cdot) = \Sigma, \quad v_{xxx}(L,\cdot) = 0 & \text{on} & (0,T), \\ v(\cdot,0) = v_0, \quad v'(\cdot,0) = v_1 & \text{in} & (0,L). \end{array} \right.$$

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Question: The function Σ drives the system to the equilibrium in time T, that is,

$$v(\cdot,T) = v'(\cdot,T) = 0 \quad \text{em} \quad]0,L[?$$

The goals in Lagnese-Lions (1988):

(i) to show that the control time T is independent of k, for any given initial state, and to find, for each k, a control Θ_k driving the M-T system to rest at time T, and

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(ii) to study the behavior of Θ_k as $k \to \infty$.

Conjecture by Lagnese-Lions: *as $k \to \infty$, Θ_k converges, in some appropriate sense, towards a control driving the Kirchhoff system to equilibrium in time T.*

Main results:

- The controls Θ_k of the M-T system may diverge exponentially as $k \to \infty$.

[3]Araruna-Zuazua (2008)

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- The controls Θ_k of the M-T system may diverge exponentially as $k \rightarrow \infty$.
- By analyzing the underlying spectrum, it is possible to decompose the adjoint M-T system into two subsystems. It is sufficient to obtain a uniform (with relation to k) observability inequality for one of these subsystems.

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- Accordingly, the exact controllability requirement on M-T system is relaxed to a partial controllability property over a suitable projection of solutions, and the controls Θ_k remain bounded as $k \to \infty$.

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Main results:

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- Accordingly, the exact controllability requirement on M-T system is relaxed to a partial controllability property over a suitable projection of solutions, and the controls Θ_k remain bounded as $k \to \infty$.
- The partial controls Θ_k obtained this way converge to an exact control for the limit Kirchhoff system.

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Adjoint M-T system

$$\left|\begin{array}{lll} \frac{\rho h^3}{12}\phi'' - \phi_{xx} + k\left(\phi + \psi_x\right) = f & \text{in} & Q, \\ \rho h \psi'' - k\left(\phi + \psi_x\right)_x = g & \text{in} & Q, \\ \phi\left(0,t\right) = \phi\left(L,t\right) = \psi_x\left(0,t\right) = \psi_x\left(L,t\right) = 0 & \text{on} & \left(0,T\right), \\ \phi\left(x,0\right) = \phi_0\left(x\right), \quad \phi'\left(x,0\right) = \phi_1\left(x\right) & \text{on} & \left(0,L\right), \\ \psi\left(x,0\right) = \psi_0\left(x\right), \quad \psi'\left(x,0\right) = \psi_1\left(x\right) & \text{in} & \left(0,L\right). \end{array}\right. \tag{1}$$

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- existence, uniqueness and regularity;

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- existence, uniqueness and regularity;
- asymptotic limit as $k \to \infty$;
- spectral analysis.

Spectral analysis

$$\Phi' = -i\mathcal{A}\Phi,$$

$$\Phi = [\phi, \phi', \psi, \psi']^T,\ \mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$$

$$\mathcal{A} = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{12}{\rho h^3}\left(\frac{\partial^2}{\partial x^2} - k\right) & 0 & -\frac{12k}{\rho h^3}\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{\rho h}\frac{\partial}{\partial x} & 0 & \frac{k}{\rho h}\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}$$

with domain

$$D(\mathcal{A}) = \left[H_0^1(0,L) \cap H^2(0,L)\right] \times H_0^1(0,L) \times W \times H^1(0,L).$$

$$\mathcal{A}\Phi = \lambda\Phi. \qquad (2)$$

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In view of the various equations involved in (2) and the boundary conditions satisfied by the components ϕ and ψ, the solutions $\Phi = [\phi, \phi', \psi, \psi']^T$ associated with the eigenfunctions are such that

$$\{\phi(x,t), \psi(x,t)\} = e^{-i\lambda t}\{\sin(m\pi x/L), c\cos(m\pi x/L)\},$$

where the constant c is to be determined in terms of m and λ.

From (2) we have

$$\phi_{xxxx} - \left(\frac{\rho h \lambda^2}{k} + \frac{\lambda^2 \rho h^3}{12}\right)\phi_{xx} + \left(\frac{\lambda^4 \rho^2 h^4}{12k} + \lambda^2 \rho h\right)\phi = 0.$$

Since $\phi(x,t) = e^{-i\lambda t}\sin(m\pi x/L)$, we obtain

$$\lambda^4 - \left(\frac{12\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{\rho h L^2} + \frac{12k}{\rho h^3}\right)\lambda^2 + \frac{12\pi^4 k m^4}{\rho^2 h^4 L^4} = 0, \qquad (3)$$

while c satisfies

$$c = \frac{\pi^3 m^3}{\lambda^2 \rho h L^3} - \frac{h^2 m\pi}{12L}. \qquad (4)$$

We find

$$\widetilde{\lambda}^{\pm}_{k,m} = \pm \left[\frac{6\pi^2 m^2}{\rho h^3 L^2} + \frac{\pi^2 k m^2}{2\rho h L^2} + \frac{6k}{\rho h^3} + \right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{144k^2}{\rho^2 h^6} + \frac{288\pi^2 k m^2}{\rho^2 h^6 L^2} + \frac{24\pi^2 k^2 m^2}{\rho^2 h^4 L^2} + \left(\frac{12\pi^2 m^2}{\rho h^3 L^2} - \frac{\pi^2 k m^2}{\rho h L^2} \right)^2} \right]^{\frac{1}{2}}$$

and

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Proposition

For fixed $m \in \mathbb{N}$, as $k \rightarrow \infty$,

$$\lambda_{k,m}^{\pm} \rightarrow \lambda_m^{\pm} = \pm\sqrt{\frac{12\pi^4 m^4}{12\rho h L^4 + \pi^2 \rho h^3 L^2 m^2}}. \tag{6}$$

These are the eigenvalues of the limit Kirchhoff system for which the corresponding eigenfunctions are $\cos(m\pi x/L)$.

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- ϕ - angle of rotation

Nonlinear Mindlin-Timoshenko system

$$\left| \begin{array}{ll} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x - \left[\psi_x \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q, \end{array} \right.$$

- $Q = (0, L) \times (0, T)$
- ϕ - angle of rotation
- ψ - vertical displacement

Nonlinear Mindlin-Timoshenko system

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- $Q = (0, L) \times (0, T)$
- ϕ - angle of rotation
- ψ - vertical displacement
- η - longitudinal displacement

Nonlinear Mindlin-Timoshenko system

$$
\left|
\begin{array}{lll}
\dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\,(\phi + \psi_x) = 0 & \text{in} & Q, \\
\rho h \psi_{tt} - k\,(\phi + \psi_x)_x - \left[\psi_x \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} & Q, \\
\rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} & Q,
\end{array}
\right.
$$

- $Q = (0, L) \times (0, T)$
- ϕ - angle of rotation
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- η - longitudinal displacement
- ρ - density, h - thickness of the beam

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- $Q = (0, L) \times (0, T)$
- ϕ - angle of rotation
- ψ - vertical displacement
- η - longitudinal displacement
- ρ - density, h - thickness of the beam
- $k > 0$ - modulus of elasticity in shear

Nonlinear Mindlin-Timoshenko system

$$\left|\begin{array}{lll}
\dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\left(\phi + \psi_x\right) = 0 & \text{in} & Q, \\
\rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} & Q, \\
\rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} & Q, \\
\phi\left(0,\cdot\right) = \phi\left(L,\cdot\right) = 0 & \text{on} & \left(0,T\right), \\
\psi\left(0,\cdot\right) = \psi\left(L,\cdot\right) = 0 & \text{on} & \left(0,T\right), \\
\eta_x\left(0,\cdot\right) = \eta_x\left(L,\cdot\right) = 0 & \text{on} & \left(0,T\right), \\
\left(\phi\left(\cdot,0\right), \psi\left(\cdot,0\right), \eta\left(\cdot,0\right)\right) = \left(\phi_0, \psi_0, \eta_0\right) & \text{in} & \left(0,L\right), \\
\left(\phi_t\left(\cdot,0\right), \psi_t\left(\cdot,0\right), \eta_t\left(\cdot,0\right)\right) = \left(\phi_1, \psi_1, \eta_1\right) & \text{in} & \left(0,L\right).
\end{array}\right. \quad (7)$$

Nonlinear Mindlin-Timoshenko system

The energy $E_k(t)$ given by

$$E_k(t) = \frac{1}{2}\left(\frac{\rho h^3}{12}\left|\phi_t(t)\right|^2 + \rho h\left|\psi_t(t)\right|^2 + \rho h\left|\eta_t(t)\right|^2 + \left|\phi_x(t)\right|^2 + k\left|\phi(t) + \psi_x(t)\right|^2 + \left|\eta_x(t) + \frac{1}{2}\left(\psi_x(t)\right)^2\right|^2\right) \quad (8)$$

satisfies

$$E_k(t) = E_k(0), \quad \forall t \geq 0.$$

von Kármán system

Assuming that the transverse shear effects are neglected, we obtain the so called von Kármán system:

$$\left|\begin{array}{lll} \rho h\psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} - \left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} & Q, \\ \rho h\eta_{tt} - \left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \text{in} & Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{on} & (0,T), \\ \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 & \text{on} & (0,T), \\ (\psi(\cdot,0), \psi_t(\cdot,0), \eta(\cdot,0), \eta_t(\cdot,0)) = (\psi_0, \psi_1, \eta_0, \eta_1) & \text{in} & (0,L), \end{array}\right. \tag{9}$$

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Assuming that the transverse shear effects are neglected, we obtain the so called von Kármán system:

$$\left| \begin{array}{lll} \rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} - \left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} & Q, \\ \rho h \eta_{tt} - \left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \text{in} & Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0 & \text{on} & (0,T), \\ \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 & \text{on} & (0,T), \\ (\psi(\cdot,0), \psi_t(\cdot,0), \eta(\cdot,0), \eta_t(\cdot,0)) = (\psi_0, \psi_1, \eta_0, \eta_1) & \text{in} & (0,L), \end{array} \right. \tag{9}$$

Neglecting the shear effects of the beam is equivalent to making $k \to \infty$ in (7).

von Kármán system

The energy $E(t)$ of (9):

$$E(t) = \frac{1}{2}\left(\rho h\left|\psi_t(t)\right|^2 + \rho h\left|\eta_t(t)\right|^2 + \frac{\rho h^3}{12}\left|\psi_{xt}(t)\right|^2 + \left|\psi_{xx}(t)\right|^2 + \left|\eta_x(t) + \frac{1}{2}\psi_x^2(t)\right|^2\right) \tag{10}$$

is conservative, that is, $E(t) = E(0)$, for all $t \in [0, T]$.

Notations

- $H = \left\{ v \in L^2(0, L) ; \int_0^L v(x)\, dx = 0 \right\}$

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equipped with the norm

$$\|(u_1, u_2, v_1, v_2, w_1, w_2)\|_k^2 = |u_{1x}|^2 + \frac{\rho h^3}{12} |u_2|^2 + k |u_1 + v_{1x}|^2 + \rho h |v_2|^2 + |w_{1x}|^2 + \rho h |w_2|^2 ,$$

where $|\cdot|$ denotes the norm in $L^2(0,L)$.

Contents

Existence and uniqueness of solution

Theorem

Let $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}$. *Then, problem* (7) *has a unique weak solution in the class*

$$\begin{aligned}(\phi, \psi, \eta) \in \quad & C^0\left([0,\infty); \left[H_0^1(0,L)\right]^2 \times V\right) \\ & \cap C^1\left([0,\infty); \left[L^2(0,L)\right]^2 \times H\right). \end{aligned} \tag{11}$$

Idea of the proof

We employ the semigroup theory. The problem (7) can be written in the form:

$$\left| \begin{array}{l} U_t = \mathcal{A}U + F(U), \\ U(0) = U_0, \end{array} \right.$$

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We employ the semigroup theory. The problem (7) can be written in the form:

$$\left| \begin{array}{l} U_t = \mathcal{A}U + F(U), \\ U(0) = U_0, \end{array} \right.$$

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{12}{\rho h^3}\left(\frac{\partial^2}{\partial x^2} - k\right) & 0 & -\frac{12k}{\rho h^3}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{\rho h}\frac{\partial}{\partial x} & 0 & \frac{k}{\rho h}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho h}\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \phi \\ \phi' \\ \psi \\ \psi' \\ \eta \\ \eta' \end{bmatrix},$$

Idea of the proof

$$F(U) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left[\psi_x \left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x \\ 0 \\ \frac{1}{2}\left(\psi_x^2\right)_x \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \\ \psi_1 \\ \eta_0 \\ \eta_1 \end{bmatrix}.$$

Idea of the proof

$$F(U) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left[\psi_x \left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x \\ 0 \\ \frac{1}{2}\left(\psi_x^2\right)_x \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \\ \psi_1 \\ \eta_0 \\ \eta_1 \end{bmatrix}.$$

- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$

Idea of the proof

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- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$

- $D(\mathcal{A}) = \left\{ \left[H_0^1(0,L) \cap H_0^1(0,L)\right] \times H_0^1(0,L) \right\}^2 \times W \times H^1(0,L)$

Idea of the proof

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- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$

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- $W = \left\{v \in H^2(0,L);\ v_x(0) = v_x(L) = 0\right\}$

Idea of the proof

- $\mathcal{A}$ is the infinitesimal generator of a semigroup in $\mathcal{X}$

Idea of the proof

- $\mathcal{A}$ is the infinitesimal generator of a semigroup in $\mathcal{X}$

- $F(U)$ is locally Lipschitz continuous in $\mathcal{X}$.

Contents

Asymptotic Limit as $k \to \infty$

Theorem

Let $\left(\phi^k, \psi^k, \eta^k\right)$ *be the unique solution of* (7) *with data* $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}$ *satisfying*

$$\phi_0 + \psi_{0x} = 0 \quad in \quad (0, L). \tag{12}$$

Then, as $k \to \infty$*, the following convergence property holds:*

$$\left\{\phi^k, \psi^k, \eta^k\right\} \to \{-\psi_x, \psi, \eta\} \text{ weak}* \text{ in } L^\infty\left(0, T; \left[H_0^1(0, L)\right]^2 \times V\right),$$

where (ψ, η) *solves the von Kármán system* (9).

[4]

[4] F. D. Araruna, P. Braz e Silva and E. Zuazua (in preparation)

Idea of the proof

For $\epsilon \in (0,1)$ fixed, we consider the perturbed system:

$$
\left|\begin{array}{ll}
\dfrac{\rho h^3}{12}\phi^\epsilon_{tt} - \phi^\epsilon_{xx} + k\left(\phi^\epsilon + \psi^\epsilon_x\right) = 0 & \text{in} \quad Q, \\
\rho h \psi^\epsilon_{tt} - k\left(\phi^\epsilon + \psi^\epsilon_x\right)_x - \left[\psi^\epsilon_x\left(\eta^\epsilon_x + \dfrac{1}{2}\left(\psi^\epsilon_x\right)^2\right)\right]_x + \epsilon\psi^\epsilon_{xxxx} = 0 & \text{in} \quad Q, \\
\rho h \eta^\epsilon_{tt} - \left(\eta^\epsilon_x + \dfrac{1}{2}\left(\psi^\epsilon_x\right)^2\right)_x = 0 & \text{in} \quad Q, \\
\phi^\epsilon(0,\cdot) = \phi^\epsilon(L,\cdot) = 0 \quad \text{on} \quad (0,T), & \\
\psi^\epsilon(0,\cdot) = \psi^\epsilon(L,\cdot) = \psi^\epsilon_x(0,\cdot) = \psi^\epsilon_x(L,\cdot) = 0 \quad \text{on} \quad (0,T), & \\
\eta^\epsilon_x(0,\cdot) = \eta^\epsilon_x(L,\cdot) = 0 \quad \text{on} \quad (0,T), & \\
\left(\phi^\epsilon(\cdot,0), \psi^\epsilon(\cdot,0), \eta^\epsilon(\cdot,0)\right) = \left(\phi_0, \psi_0, \eta_0\right) \quad \text{in} \quad (0,L), & \\
\left(\psi^\epsilon_t(\cdot,0), \phi^\epsilon_t(\cdot,0), \eta^\epsilon_t(\cdot,0)\right) = \left(\phi_1, \psi_1, \eta_1\right) \quad \text{in} \quad (0,L). &
\end{array}\right.
\tag{13}
$$

Idea of the proof

System (13) is well-posed in the energy space

$$\mathcal{X}_1 = H_0^1(0, L) \times L^2(0, L) \times H_0^2(0, L) \times L^2(0, L) \times V \times H,$$

Idea of the proof

System (13) is well-posed in the energy space

$$\mathcal{X}_1 = H_0^1(0,L) \times L^2(0,L) \times H_0^2(0,L) \times L^2(0,L) \times V \times H,$$

that is, for any $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}_1$, there exists a unique solution in the class

$$\begin{aligned}(\phi^\epsilon, \psi^\epsilon, \eta^\epsilon) \in \quad & C^0\left([0,T]; H_0^1(0,L) \times H_0^2(0,L) \times V\right) \\ & \cap C^1\left([0,T]; \left[L^2(0,L)\right]^2 \times H\right).\end{aligned}$$

Idea of the proof

For each k, let $(\phi^{\epsilon,k}, \psi^{\epsilon,k}, \eta^{\epsilon,k})$ be the solution of system (13) with data $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}_1$.

Idea of the proof

For each k, let $(\phi^{\epsilon,k}, \psi^{\epsilon,k}, \eta^{\epsilon,k})$ be the solution of system (13) with data $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}_1$.
The energy of system (13):

$$E_{\epsilon,k}(t) = \frac{1}{2}\Bigg(\frac{\rho h^3}{12}\left|\phi_t^{\epsilon,k}(t)\right|^2 + \rho h\left|\psi_t^{\epsilon,k}(t)\right|^2 + \rho h\left|\eta_t^{\epsilon,k}(t)\right|^2 + \left|\phi_x^{\epsilon,k}(t)\right|^2 + k\left|\phi^{\epsilon,k}(t) + \psi_x^{\epsilon,k}(t)\right|^2 + \left|\eta_x^{\epsilon,k}(t) + \frac{1}{2}\left(\psi_x^{\epsilon,k}(t)\right)^2\right|^2 + \epsilon\left|\psi_{xx}^{\epsilon,k}(t)\right|^2\Bigg). \tag{14}$$

satisfies

$$E_{\epsilon,k}(t) = E_{\epsilon,k}(0), \ \forall t \in [0, T]. \tag{15}$$

Idea of the proof

Initially we want to show that system (13) approaches, as $k \to \infty$, the modified von Kármán system:

$$\left| \begin{array}{l} \rho h \psi^{\epsilon}_{tt} - \frac{\rho h^3}{12} \psi^{\epsilon}_{xxtt} + (1+\epsilon)\, \psi^{\epsilon}_{xxxx} - \left[\psi^{\epsilon}_{x} \left(\eta^{\epsilon}_{x} + \frac{1}{2} (\psi^{\epsilon}_{x})^2 \right) \right]_x = 0 \\ \rho h \eta^{\epsilon}_{tt} - \left(\eta^{\epsilon}_{x} + \frac{1}{2} (\psi^{\epsilon}_{x})^{\epsilon} \right)_x = 0 \\ \psi^{\epsilon}(0,\cdot) = \psi^{\epsilon}(L,\cdot) = \psi^{\epsilon}_{x}(0,\cdot) = \psi^{\epsilon}_{x}(L,\cdot) = 0 \\ \eta^{\epsilon}_{x}(0,\cdot) = \eta^{\epsilon}_{x}(L,\cdot) = 0 \\ (\psi^{\epsilon}(\cdot,0), \psi^{\epsilon}_{t}(\cdot,0), \eta^{\epsilon}(\cdot,0), \eta^{\epsilon}_{t}(\cdot,0)) = (\psi_0, \psi_1, \eta_0, \eta_1) \end{array} \right. \tag{16}$$

Idea of the proof

In fact, considering the initial data $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1)$ satisfying (12) , we obtain

$$E_{\epsilon,k}(0) \leq C, \ \forall\, k > 0, \ \forall \epsilon \in (0,1)\,.$$

Idea of the proof

In fact, considering the initial data $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1)$ satisfying (12), we obtain

$$E_{\epsilon,k}(0) \leq C, \ \forall\, k > 0, \ \forall \epsilon \in (0,1).$$

In this way, we can deduce that the sequences (in k)

$$\left\{\phi^{\epsilon,k}\right\}, \ \left\{\psi^{\epsilon,k}\right\}, \ \left\{\eta^{\epsilon,k}\right\}$$

are bounded in $L^\infty\left(0, T; H_0^1(0,L)\right)$, $L^\infty\left(0, T; H_0^2(0,L)\right)$ and $L^\infty(0, T; V)$, respectively,

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In fact, considering the initial data $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1)$ satisfying (12), we obtain

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In this way, we can deduce that the sequences (in k)

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are bounded in $L^\infty\left(0, T; H_0^1(0, L)\right)$, $L^\infty\left(0, T; H_0^2(0, L)\right)$ and $L^\infty(0, T; V)$, respectively,
and

$$\left\{\phi_t^{\epsilon,k}\right\}, \ \left\{\psi_t^{\epsilon,k}\right\}, \ \left\{\eta_t^{\epsilon,k}\right\}, \ \left\{\sqrt{k}\left(\phi^{\epsilon,k} + \psi_x^{\epsilon,k}\right)\right\}, \ \left\{\eta_x^{\epsilon,k} + \frac{1}{2}\left(\psi_x^{\epsilon,k}\right)^2\right\}$$

remain bounded in $L^\infty\left(0, T; L^2(0, L)\right)$.

Idea of the proof

Extracting subsequences, we can conclude that

$$\left\{\phi^{\epsilon,k}, \psi^{\epsilon,k}, \eta^{\epsilon,k}\right\} \rightarrow \{\phi^{\epsilon}, \psi^{\epsilon}, \eta^{\epsilon}\} \text{ weak}* \text{ in } L^{\infty}\left(0, T; H_0^1 \times H_0^2 \times V\right) \tag{17}$$

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$$\phi^{\epsilon} + \psi_x^{\epsilon} = 0. \tag{18}$$

Idea of the proof

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$$\phi^{\epsilon}+\psi_x^{\epsilon}=0. \tag{18}$$

$$\left\{\phi_t^{\epsilon,k},\psi_t^{\epsilon,k},\eta_t^{\epsilon,k}\right\} \rightarrow \{\phi_t^{\epsilon},\psi_t^{\epsilon},\eta_t^{\epsilon}\} \text{ weak} * \text{ in } L^{\infty}\left(0,T;\left[L^2\left(0,L\right)\right]^3\right) \tag{19}$$

Idea of the proof

Extracting subsequences, we can conclude that

$$\left\{\phi^{\epsilon,k}, \psi^{\epsilon,k}, \eta^{\epsilon,k}\right\} \rightarrow \{\phi^{\epsilon}, \psi^{\epsilon}, \eta^{\epsilon}\} \text{ weak}* \text{ in } L^{\infty}\left(0, T; H_0^1 \times H_0^2 \times V\right) \tag{17}$$

$$\phi^{\epsilon} + \psi_x^{\epsilon} = 0. \tag{18}$$

$$\left\{\phi_t^{\epsilon,k}, \psi_t^{\epsilon,k}, \eta_t^{\epsilon,k}\right\} \rightarrow \{\phi_t^{\epsilon}, \psi_t^{\epsilon}, \eta_t^{\epsilon}\} \text{ weak} * \text{ in } L^{\infty}\left(0, T; \left[L^2(0, L)\right]^3\right) \tag{19}$$

$$\eta_x^{\epsilon,k} + \frac{1}{2}\left(\psi_x^{\epsilon,k}\right)^2 \rightarrow \xi \text{ weak} - * \text{ in } L^{\infty}\left(0, T; L^2(0, L)\right). \tag{20}$$

Idea of the proof

By the uniform bounded of $\left\{\psi^{\epsilon,k}\right\}$ in $L^{\infty}\left(0,T;H_0^2\left(0,L\right)\right)$ and Aubin-Lions compactness theorem, we get

$$\psi^{\epsilon,k} \longrightarrow \psi^{\epsilon} \text{ strongly in } L^{\infty}\left(0,T;H^{2-\delta}\left(0,L\right)\right), \tag{21}$$

for any $\delta > 0$.

Idea of the proof

By the uniform bounded of $\{\psi^{\epsilon,k}\}$ in $L^\infty(0, T; H_0^2(0, L))$ and Aubin-Lions compactness theorem, we get

$$\psi^{\epsilon,k} \to \psi^\epsilon \text{ strongly in } L^\infty\left(0, T; H^{2-\delta}(0, L)\right), \tag{21}$$

for any $\delta > 0$.

Combining (17) and (21), it result that $\xi = \eta_x^\epsilon + (\psi_x^\epsilon)^2/2$ and

$$\psi_x^{\epsilon,k}\left[\eta_x^{\epsilon,k} + \frac{1}{2}\left(\psi_x^{\epsilon,k}\right)^2\right] \to \psi_x^\epsilon\left[\eta_x^\epsilon + \frac{1}{2}(\psi_x^\epsilon)^2\right] \text{ weakly in } L^2(Q). \tag{22}$$

Idea of the proof

We consider now the energy of the system (16):

$$E_\epsilon(t) = \frac{1}{2}\left(\rho h\left|\psi_t^\epsilon(t)\right|^2 + \rho h\left|\eta_t^\epsilon(t)\right|^2 + \frac{\rho h^3}{12}\left|\psi_{xt}^\epsilon(t)\right|^2 + (1+\epsilon)\left|\psi_{xx}^\epsilon(t)\right|^2 + \left|\eta_x^\epsilon(t) + \frac{1}{2}\left(\psi_x^\epsilon(t)\right)^2\right|^2\right). \tag{23}$$

Idea of the proof

We consider now the energy of the system (16):

$$E_\epsilon(t) = \frac{1}{2}\left(\rho h\left|\psi_t^\epsilon(t)\right|^2 + \rho h\left|\eta_t^\epsilon(t)\right|^2 + \frac{\rho h^3}{12}\left|\psi_{xt}^\epsilon(t)\right|^2 + (1+\epsilon)\left|\psi_{xx}^\epsilon(t)\right|^2 + \left|\eta_x^\epsilon(t) + \frac{1}{2}\left(\psi_x^\epsilon(t)\right)^2\right|^2\right). \tag{23}$$

The easy see that

$$E_\epsilon(t) = E_\epsilon(0),\ \forall\, t \in [0, T].$$

Idea of the proof

In this way, the following sequences (in ϵ) remain bounded in $L^\infty\left(0, T; L^2\left(0, L\right)\right)$:

$$\left\{\psi_t^\epsilon\right\}, \ \left\{\eta_t^\epsilon\right\}, \ \left\{\psi_{xt}^\epsilon\right\}, \ \left\{\eta_x^\epsilon\right\}, \ \left\{\psi_{xx}^\epsilon\right\}, \ \left\{\sqrt{\epsilon}\psi_{xx}^\epsilon\right\}, \ \left\{\eta_x^\epsilon + \frac{1}{2}\left(\psi_x^\epsilon\right)^2\right\}.$$

Idea of the proof

In this way, the following sequences (in ϵ) remain bounded in $L^{\infty}\left(0, T; L^{2}\left(0, L\right)\right)$:

$$\left\{\psi_{t}^{\epsilon}\right\},\ \left\{\eta_{t}^{\epsilon}\right\},\ \left\{\psi_{xt}^{\epsilon}\right\},\ \left\{\eta_{x}^{\epsilon}\right\},\ \left\{\psi_{xx}^{\epsilon}\right\},\ \left\{\sqrt{\epsilon}\psi_{xx}^{\epsilon}\right\},\ \left\{\eta_{x}^{\epsilon}+\frac{1}{2}\left(\psi_{x}^{\epsilon}\right)^{2}\right\}.$$

Extracting subsequences, we deduce that

$$\left\{\psi^{\epsilon},\sqrt{\epsilon}\psi^{\epsilon},\eta^{\epsilon}\right\} \rightarrow \left\{\phi,\alpha,\eta\right\} \text{ weak} * \text{ in } L^{\infty}\left(0, T; \left[H_{0}^{2}\left(0, L\right)\right]^{2} \times V\right)$$

Idea of the proof

In this way, the following sequences (in ϵ) remain bounded in $L^\infty\left(0, T; L^2(0, L)\right)$:

$$\{\psi_t^\epsilon\},\ \{\eta_t^\epsilon\},\ \{\psi_{xt}^\epsilon\},\ \{\eta_x^\epsilon\},\ \{\psi_{xx}^\epsilon\},\ \left\{\sqrt{\epsilon}\psi_{xx}^\epsilon\right\},\ \left\{\eta_x^\epsilon + \frac{1}{2}(\psi_x^\epsilon)^2\right\}.$$

Extracting subsequences, we deduce that

$$\left\{\psi^\epsilon, \sqrt{\epsilon}\psi^\epsilon, \eta^\epsilon\right\} \rightarrow \{\phi, \alpha, \eta\} \text{ weak} * \text{ in } L^\infty\left(0, T; \left[H_0^2(0, L)\right]^2 \times V\right)$$

$$\{\psi_t^\epsilon, \eta_t^\epsilon\} \rightarrow \{\psi_t, \eta_t\} \text{ weak} * \text{ in } L^\infty\left(0, T; H_0^1(0, L) \times L^2(0, L)\right)$$

Idea of the proof

In this way, the following sequences (in ϵ) remain bounded in $L^\infty\left(0, T; L^2(0, L)\right)$:

$$\{\psi_t^\epsilon\},\ \{\eta_t^\epsilon\},\ \{\psi_{xt}^\epsilon\},\ \{\eta_x^\epsilon\},\ \{\psi_{xx}^\epsilon\},\ \{\sqrt{\epsilon}\psi_{xx}^\epsilon\},\ \left\{\eta_x^\epsilon + \frac{1}{2}(\psi_x^\epsilon)^2\right\}.$$

Extracting subsequences, we deduce that

$$\left\{\psi^\epsilon, \sqrt{\epsilon}\psi^\epsilon, \eta^\epsilon\right\} \rightarrow \{\phi, \alpha, \eta\} \text{ weak} * \text{ in } L^\infty\left(0, T; \left[H_0^2(0, L)\right]^2 \times V\right)$$

$$\{\psi_t^\epsilon, \eta_t^\epsilon\} \rightarrow \{\psi_t, \eta_t\} \text{ weak} * \text{ in } L^\infty\left(0, T; H_0^1(0, L) \times L^2(0, L)\right)$$

$$\eta_x^\epsilon + \frac{1}{2}(\psi_x^\epsilon)^2 \rightarrow \eta_x + \frac{1}{2}(\psi_x)^2 \text{ weak} * \text{ in } L^\infty\left(0, T; L^2(0, L)\right)$$

Contents

Uniform Stabilization as $k \longrightarrow \infty$

The aim is to obtain the exponential decay for the energy (10) associated to solution of the von Kármán system

$$\left| \begin{array}{l} \rho h\psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} - \left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \psi_t - \psi_{xxt} = 0, \\ \rho h\eta_{tt} - \left(\eta_x + \frac{1}{2}\psi_x^2\right)_x + \eta_t = 0, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = 0, \\ \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0, \\ (\psi(\cdot,0), \psi_t(\cdot,0), \eta(\cdot,0), \eta_t(\cdot,0)) = (\psi_0, \psi_1, \eta_0, \eta_1) \end{array} \right. \qquad (24)$$

Uniform Stabilization as $k \to \infty$

as limit (as $k \to \infty$) of the uniform stabilization of the Mindlin-Timoshenko one

$$
\left|
\begin{array}{lll}
\dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\left(\phi + \psi_x\right) + \phi_t = 0, & \text{in} & Q, \\
\rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x + \psi_t = 0, & \text{in} & Q, \\
\rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x + \eta_t = 0, & \text{in} & Q, \\
\phi\left(0,\cdot\right) = \phi\left(L,\cdot\right) = 0, & \text{on} & \left(0,T\right), \\
\psi\left(0,\cdot\right) = \psi\left(L,\cdot\right) = 0 & \text{on} & \left(0,T\right), \\
\eta_x\left(0,\cdot\right) = \eta_x\left(L,\cdot\right) = 0 & \text{on} & \left(0,T\right), \\
\left(\phi\left(\cdot,0\right), \psi\left(\cdot,0\right), \eta\left(\cdot,0\right)\right) = \left(\phi_0, \psi_0, \eta_0\right) & \text{in} & \left(0,L\right), \\
\left(\phi_t\left(\cdot,0\right), \psi_t\left(\cdot,0\right), \eta_t\left(\cdot,0\right)\right) = \left(\phi_1, \psi_1, \eta_1\right) & \text{in} & \left(0,L\right).
\end{array}
\right. \tag{25}
$$

Uniform Stabilization as $k \longrightarrow \infty$

Theorem

Let (ϕ, ψ, η) *be the global solution of* (25) *for data* $(\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1) \in \mathcal{X}$. *Then there exists a constant* $\omega > 0$ *such that*

$$E_k(t) \leq 4E_k(0)\, e^{-\frac{\omega}{2}t},\ \forall t \geq 0. \tag{26}$$

Thank you!