An Asymptotic Preserving Scheme for the Kolmogorov Equation

Erich L Foster

basque center for applied mathematics

09 May 2014

Kolmogorov Equation

- The Kolmogorov equation is a simplified version of the Fokker-Planck equation and is given by

$$\partial_t f - \partial_v^2 f - v\,\partial_x f = 0 \qquad (t, v, x) \in \mathbb{R}_+^* \times \mathbb{R}^2 \qquad (1)$$

- At a quick glance it appears diffusion will only occur in v-direction and transport in the x-direction.
- The $v\,\partial_x$ actually creates a complex interaction where things will appear to rotate allowing diffusion along "spirals."

Kolmogorov Equation

- The Kolmogorov equation is a simplified version of the Fokker-Planck equation and is given by

$$\partial_t f - \partial_v^2 f - v\,\partial_x f = 0 \qquad (t, v, x) \in \mathbb{R}_+^* \times \mathbb{R}^2 \qquad (1)$$

- At a quick glance it appears diffusion will only occur in v-direction and transport in the x-direction.
- The $v\,\partial_x$ actually creates a complex interaction where things will appear to rotate allowing diffusion along "spirals."

Kolmogorov Equation

- The Kolmogorov equation is a simplified version of the Fokker-Planck equation and is given by

$$\partial_t f - \partial_v^2 f - v\,\partial_x f = 0 \qquad (t, v, x) \in \mathbb{R}_+^* \times \mathbb{R}^2 \qquad (1)$$

- At a quick glance it appears diffusion will only occur in v-direction and transport in the x-direction.
- The $v\,\partial_x$ actually creates a complex interaction where things will appear to rotate allowing diffusion along "spirals."

Numerical Niceties

- Can obtain an exact solution.

Numerical Difficulties

- Numerical simulation, such as FEM, requires artificial boundary conditions.
- Highly convective and needs a way to stabilize numerical schemes.
- Long time integration requires large domain and thus is expensive.

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Possible Numerical Approaches

- Use operator splitting.
  - Operators don't commute, thus added error.
  - Requires large domain and artificial boundary conditions.
- Stabilization scheme.
  - Adds diffusion and thus changes decay rate.
  - Requires large domain and artificial boundary conditions.
- Change of Variables

Rotating Frame

- Make the change of variables

$$g(t, v, z) = f(t, v, x + t\,v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\,\partial_{vz}^2 g + t^2\,\partial_z^2 g \qquad (2)$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Rotating Frame

- Make the change of variables

$$g(t, v, z) = f(t, v, x + t\, v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\, \partial_{vz}^2 g + t^2\, \partial_z^2 g \tag{2}$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Rotating Frame

- Make the change of variables

$$g(t, v, z) = f(t, v, x + t\,v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\,\partial_{vz}^2 g + t^2\,\partial_z^2 g \tag{2}$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Rotating Frame

- Make the change of variables

$$g(t,v,z) = f(t,v,x+t\,v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\,\partial_{vz}^2 g + t^2\,\partial_z^2 g \tag{2}$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Rotating Frame

- Make the change of variables

$$g(t, v, z) = f(t, v, x + t\,v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\,\partial_{vz}^2 g + t^2\,\partial_z^2 g \tag{2}$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Rotating Frame

- Make the change of variables

$$g(t, v, z) = f(t, v, x + t\,v)$$

- Then the Kolmogorov equation, (1) in the rotating frame becomes

$$\partial_t g = \partial_v^2 g + 2t\,\partial_{vz}^2 g + t^2\,\partial_z^2 g \qquad (2)$$

- Diffusion is clearly seen in the direction of z.
- Stability is no-longer an issue.
- Still need artificial boundary conditions.
- Still need large domain for long time integration.

Similarity Variables

- Using the change of variables

$$h(s,\tilde{v},\tilde{x}) = e^{2\,s} g(e^{s}-1, e^{s/2}\tilde{v}, e^{3\,s/2}\tilde{x})$$

the Kolmogorov equation, (1), becomes the following

$$\partial_s h = 2h + \partial_v^2 h + 2\,A(s)\,\partial_{vx} h + A^2(s)\,\partial_x^2 h + \frac{1}{2}\,v\,\partial_v h + \frac{3}{2}\,x\,\partial_x h,$$

where $A(s) = 1 - e^{-s}$.

- Still need artificial boundary conditions.
- Can use small domain for long time integration.

Similarity Variables

- Using the change of variables

$$h(s,\tilde{v},\tilde{x}) = e^{2\,s} g(e^{s}-1, e^{s/2}\tilde{v}, e^{3\,s/2}\tilde{x})$$

the Kolmogorov equation, (1), becomes the following

$$\partial_s h = 2h + \partial_v^2 h + 2\,A(s)\,\partial_{vx} h + A^2(s)\,\partial_x^2 h + \frac{1}{2}\,v\,\partial_v h + \frac{3}{2}\,x\,\partial_x h,$$

where $A(s) = 1 - e^{-s}$.

- Still need artificial boundary conditions.
- Can use small domain for long time integration.

Similarity Variables

- Using the change of variables

$$h(s,\tilde{v},\tilde{x}) = e^{2\,s} g(e^{s}-1, e^{s/2}\tilde{v}, e^{3\,s/2}\tilde{x})$$

the Kolmogorov equation, (1), becomes the following

$$\partial_s h = 2h + \partial_v^2 h + 2\,A(s)\,\partial_{vx} h + A^2(s)\,\partial_x^2 h + \frac{1}{2}\,v\,\partial_v h + \frac{3}{2}\,x\,\partial_x h,$$

where $A(s) = 1 - e^{-s}$.

- Still need artificial boundary conditions.
- Can use small domain for long time integration.

Exact Solution

Use Fourier transform and solve resulting ODE.

Rotating Frame

$$g(t, v, z) = (g_0 * G_t)(v, z), \tag{3}$$

where

$$G_t(v, z) = \frac{\sqrt{3}}{2\pi t^2} e^{-\frac{1}{4t^3}(3z^2+(2t\,v-3z)^2)}. \tag{4}$$

Similarity Variables

$$h(s, \tilde{v}, \tilde{x}) = e^{2s}(g_0 * G_{e^s-1})(e^{s/2}\tilde{v}, e^{3s/2}\tilde{x}). \tag{5}$$

Exact Solution

Use Fourier transform and solve resulting ODE.

Rotating Frame

$$g(t, v, z) = (g_0 * G_t)(v, z), \tag{3}$$

where

$$G_t(v, z) = \frac{\sqrt{3}}{2\pi t^2} e^{-\frac{1}{4t^3}(3z^2 + (2t\,v - 3z)^2)}. \tag{4}$$

Similarity Variables

$$h(s, \tilde{v}, \tilde{x}) = e^{2s}(g_0 * G_{e^s - 1})(e^{s/2}\tilde{v}, e^{3s/2}\tilde{x}). \tag{5}$$

Exact Solution

Use Fourier transform and solve resulting ODE.

Rotating Frame

$$g(t, v, z) = (g_0 * G_t)(v, z), \tag{3}$$

where

$$G_t(v, z) = \frac{\sqrt{3}}{2\pi t^2} e^{-\frac{1}{4t^3}(3z^2+(2t\,v-3z)^2)}. \tag{4}$$

Similarity Variables

$$h(s, \tilde{v}, \tilde{x}) = e^{2s}(g_0 * G_{e^s-1})(e^{s/2}\tilde{v}, e^{3s/2}\tilde{x}). \tag{5}$$

Long Time Behavior

Taking the initial condition to be

$$f_0(v,x) = e^{-(v^2+x^2)}. \qquad (6)$$

Then the solution (3) as $t \to \infty$ is

$$\lim_{t\to\infty} g(t,\tilde{v},\tilde{x}) = 0.$$

Whereas the solution (5) as $s \to \infty$ is

$$\lim_{s\to\infty} h(s,\tilde{v},\tilde{x}) = \frac{\sqrt{3}}{2} G_1(\tilde{v},\tilde{x}). \qquad (7)$$

Long Time Behavior

Taking the initial condition to be

$$f_0(v,x) = e^{-(v^2+x^2)}. \tag{6}$$

Then the solution (3) as $t \to \infty$ is

$$\lim_{t\to\infty} g(t,\tilde{v},\tilde{x}) = 0.$$

Whereas the solution (5) as $s \to \infty$ is

$$\lim_{s\to\infty} h(s,\tilde{v},\tilde{x}) = \frac{\sqrt{3}}{2} G_1(\tilde{v},\tilde{x}). \tag{7}$$

Long Time Behavior

Taking the initial condition to be

$$f_0(v,x) = e^{-(v^2+x^2)}. \tag{6}$$

Then the solution (3) as $t \to \infty$ is

$$\lim_{t\to\infty} g(t,\tilde{v},\tilde{x}) = 0.$$

Whereas the solution (5) as $s \to \infty$ is

$$\lim_{s\to\infty} h(s,\tilde{v},\tilde{x}) = \frac{\sqrt{3}}{2} G_1(\tilde{v},\tilde{x}). \tag{7}$$

Long Time Behavior

Taking the initial condition to be

$$f_0(v,x) = e^{-(v^2+x^2)}. \tag{6}$$

Then the solution (3) as $t \to \infty$ is

$$\lim_{t\to\infty} g(t,\tilde{v},\tilde{x}) = 0.$$

Whereas the solution (5) as $s \to \infty$ is

$$\lim_{s\to\infty} h(s,\tilde{v},\tilde{x}) = \frac{\sqrt{3}}{2} G_1(\tilde{v},\tilde{x}). \tag{7}$$

Benefits of Similarity Variables

- Small space domain.
- Fast marching in time.
- Exponential convergence to steady state.

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01.$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

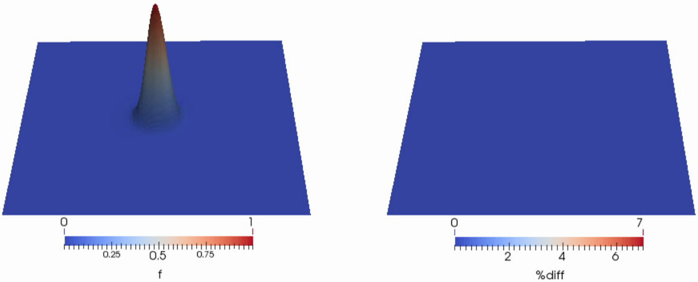
$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01.$$

Comparing Methods

- We present 5 different finite element (FE) solutions of the Kolmogorov equation
  1. A naïve FE solution.
  2. Operator splitting.
  3. Least square stabilization.
  4. Rotating frame Kolmogorov equation.
  5. Similarity Kolmogorov equation.
- Use homogeneous Dirichlet boundary conditions.
- For all simulations we take

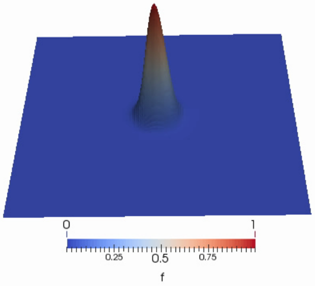
$$\Omega = [-10, 10]^2, \quad t \in [0, 10], \quad N = 128, \quad dt = 0.01.$$

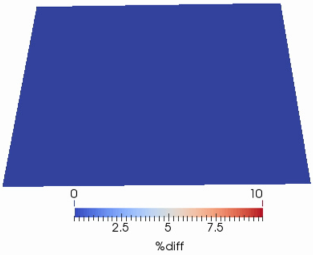
Kolmogorov Equation Naïve FE



Total Simulation time 608.844 seconds.

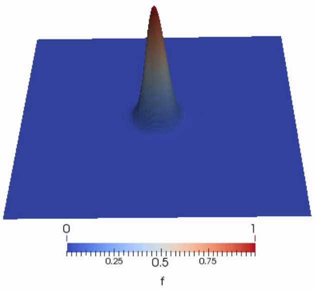
Kolmogorov Equation Operator Splitting





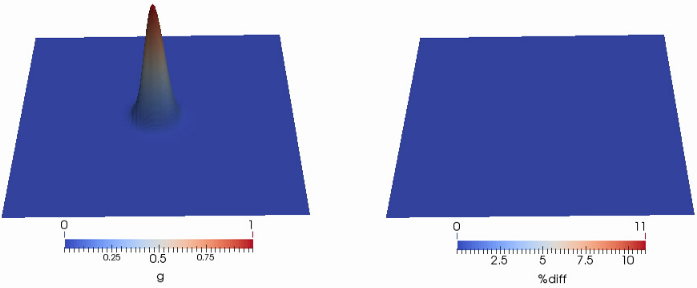
Total Simulation time 1828.52 seconds.

Kolmogorov Equation Stabilized FE



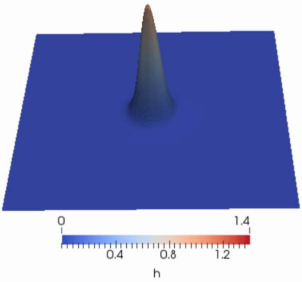
Total Simulation time 1425.34 seconds.

Rotating Kolmogorov Equation



Total Simulation time 609.757 seconds.

Similarity Kolmogorov Equation



Total Simulation time 824.909 seconds.