Time optimal control for wave-type systems

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Problem Statement

Let $X$ and $U$ be two Banach spaces and $F : X \times U \rightarrow X$. Consider the control system:

$$\dot{z} = F(z, u).$$

• Given $z_0, z_1 \in X$, $z_0 \neq z_1$, show that there exist a minimal time $\tau = \tau(z_0, z_1) > 0$ such that there exists a control $u \in L^2([0, \tau], U)$, with

$$\|u\|_{L^2([0, \tau], U)} \leq 1,$$

for which the solution of the Cauchy problem:

$$\dot{z} = F(z, u), \quad z(0) = z_0,$$

satisfies:

$$z(T) = z_1;$$

• Derive optimality conditions (Pontryagin’s maximum principle);
• Prove the saturation property, i.e. $\|u\|_{L^2([0, \tau], U)} = 1$. 
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2 Extension of the Pontryagin’s maximum principle

3 Finite dimensional spaces approximation

4 Conclusion and open questions
1. Introduction

2. Extension of the Pontryagin’s maximum principle

3. Finite dimensional spaces approximation

4. Conclusion and open questions
Let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator and let $B \in \mathcal{L}(U, X)$ be a bounded control operator. Then $A$ generate a strongly continuous group of isometries $\mathcal{T} = (\mathcal{T}_t)_{t \in \mathbb{R}}$. We consider the dynamical system described by the equation:

$$\dot{z} = Az + Bu, \quad z(0) = z_0,$$

with $u \in L^2([0, T], U)$. Then the solution of (1) writes:

$$z(t) = \mathcal{T}_t z_0 + \Phi_t u \quad (t \in [0, T]),$$

where the maps $\Phi_t \in \mathcal{L}(L^2([0, t], U), X)$ are the input to state maps:

$$\Phi_t u = \int_0^t \mathcal{T}_{t-s} Bu(s) \, ds \quad (u \in L^2([0, t], U)).$$
Let recall some classical definitions:

**Definition**

*The pair $(A, B)$ is said:*

- *approximatively controllable in time $\tau$* if $\text{Ran} \Phi_{\tau}$ is dense in $X$;
- *exactly controllable in time $\tau$* if $\text{Ran} \Phi_{\tau} = X$. 


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2 Extension of the Pontryagin’s maximum principle
   - The reachable set
   - Optimality conditions
   - Application to the wave equation

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The $L^2$ reachable set $I$

Let $R^2_t = \Phi_t(L^2([0, t], U))$. We endowed $R^2_t$ with the norm:

$$\|z\|_{R^2_t} = \inf \{\|u\|_{L^2([0,t], U)}, \Phi_t u = z\}.$$ 

Let denote the closed unit ball of $R^2_t$:

$$B^2_t(1) = \{\Phi_t u, \ u \in L^2([0, t], U), \|u\|_{L^2([0,t], U)} \leq 1\}.$$ 

The time optimal control problem writes:

**Problem (P)**

*Does the set $\{T > 0, \ z_1 - Tz_0 \in B^2_T(1)\} \text{ admits a minimum?}$*
The $L^2$ reachable set II

**Proposition**

Let $0 \leq \sigma \leq t$. Then we have the continuous inclusions:

$$R^2_\sigma \subset R^2_t \subset X.$$ 

If there exists $T_* \geq 0$ such that $R^2_t = X$ for every $t > T_*$, then the norms $\| \cdot \|_X$ and $\| \cdot \|_{R^2_t}$ are equivalent and the *control cost* in time $t > 0$ is defined by:

$$C_t = \sup_{z \neq 0} \frac{\|z\|_{R^2_t}}{\|z\|_X} \quad (t > T_*).$$

We can see that $t \mapsto C_t$ is non increasing.
Existence of the optimal time

Proposition

Let $z_0, z_1 \in X$, $z_0 \neq z_1$ such that there exists $t \geq 0$ with:

$$z_1 - T_t z_0 \in B_t^2(1).$$

Then there exist $\tau > 0$ such that:

$$\tau = \tau(z_0, z_1) = \min \{ t > 0, \ z_1 - T_t z_0 \in B_t^2(1) \}.$$
Optimality conditions

Theorem

Let \((A, B)\) be exactly controllable in any time \(T > T_* \geq 0\) and \(z_0, z_1 \in X,\) \(z_0 \neq z_1,\) such that there exists \(t > 0\) and \(u \in L^2([0, t], U)\) with \(\|u\|_{L^2([0, t], U)} \leq 1\) steering \(z_0\) to \(z_1\) in time \(t.\)

Then there exists a minimal time \(\tau = \tau(z_0, z_1) > 0\) and a time optimal control \(u^* \in L^2([0, \tau], U),\) with \(\|u^*\|_{L^2([0, \tau], U)} \leq 1\) steering \(z_0\) to \(z_1\) in time \(\tau.\)

Moreover, if \(\tau > T_*\), then

1. \(\exists \eta \in X \setminus \{0\} \text{ s.t. } u^* = \Phi_{\tau}^* \eta;\)
2. \(\|u^*\|_{L^2([0, \tau], U)} = 1;\)
3. \(u^* \text{ is unique.}\)
Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of $\tau$ and $u^*$ is ensured by the previous proposition. If $\tau > T_*$, the sketch of the proof is:
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If $\tau > T_*$, the sketch of the proof is:

1. $R^2_t = X$ for every $t > T_*$

$(A, B)$ is exactly controllable in any time $t > T_*$. 
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If \( \tau > T_* \), the sketch of the proof is:

1. \( R_t^2 = X \) for every \( t > T_* \)
2. \( z_1 - T_\tau z_0 \in \partial B^2_\tau(1) \)
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1. $R_t^2 = X$ for every $t > T_*$
2. $z_1 - T_\tau z_0 \in \partial B_\tau^2(1)$

- Assume by contradiction that $z_1 - T_\tau z_0$ is in the interior of $B_\tau^2(1)$, i.e. there exists $u \in L^2([0, \tau], U)$ such that $\|u\|_{L^2([0,\tau], U)} = r < 1$ and $z_1 - T_\tau z_0 = \Phi_\tau u$.
- Take $t \in (T_*, \tau)$ then $z_1 - T_t z_0 = \Phi_t (u1_{[0,t]}) + \varphi(\tau, t)$ with $\lim_{t \to \tau} \varphi(\tau, t) = 0$.
- Since $t \mapsto C_t$ is non increasing, there exists $t \in [T_*, \tau]$ with $\tau - t$ small enough such that there exists $\tilde{u} \in L^2([0, t], U)$, $\|\tilde{u}\| \leq 1 - r$ and $\Phi_t \tilde{u} = \varphi(\tau, t)$.
- Consequently, $u + \tilde{u}$ is a control in time $t < \tau$, with $\|u + \tilde{u}\|_{L^2([0,t], U)} \leq 1$. 
Idea of the proof

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1. \( R_t^2 = X \) for every \( t > T_* \)
2. \( z_1 - T_\tau z_0 \in \partial B^2_T(1) \)
3. \( \exists \tilde{\eta} \in X \setminus \{0\}, \langle u^*, \Phi^* \tilde{\eta} \rangle \geq \sup_{v \in L^2([0,\tau],U) \atop \|v\|_{L^2([0,\tau],U)} \leq 1} \langle v, \Phi^* \tilde{\eta} \rangle \)
Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of $\tau$ and $u^*$ is ensured by the previous proposition.
If $\tau > T_*$, the sketch of the proof is:

1. $R^2_t = \mathcal{X}$ for every $t > T_*$
2. $z_1 - T_\tau z_0 \in \partial B^2_T(1)$
3. $\exists \tilde{\eta} \in \mathcal{X} \setminus \{0\}, \langle u^*, \Phi^* \tilde{\eta} \rangle \geq \sup_{\nu \in L^2([0,\tau], U), \|
u\|_{L^2([0,\tau], U)} \leq 1} \langle \nu, \Phi^* \tilde{\eta} \rangle$

- $B^2_T(1)$ is a convex set with non empty interior.
- $\exists \tilde{\eta} \in \mathcal{X} \setminus \{0\}, \langle z_1 - T_\tau z_0, \tilde{\eta} \rangle \geq \sup_{z \in B^2_T(1)} \langle z, \tilde{\eta} \rangle$
Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of $\tau$ and $u^\star$ is ensured by the previous proposition.

If $\tau > T^\star$, the sketch of the proof is:

1. $R^2_t = X$ for every $t > T^\star$
2. $z_1 - T^{\tau}z_0 \in \partial B^2_T(1)$
3. $\exists \tilde{\eta} \in X \setminus \{0\}, \langle u^\star, \Phi^{*\tau}\tilde{\eta} \rangle \geq \sup_{v \in L^2([0,\tau], U)} \|v\|_{L^2([0,\tau], U)} \leq 1$
4. $\exists \eta \in X \setminus \{0\}, u^\star = \Phi^{*\tau}\eta \text{ and } \|u^\star\|_{L^2([0,\tau], U)} = 1$
Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of $\tau$ and $u^*$ is ensured by the previous proposition.

If $\tau > T_*$, the sketch of the proof is:

1. $R_t^2 = X$ for every $t > T_*$
2. $z_1 - T_\tau z_0 \in \partial B^2_T$
3. $\exists \tilde{\eta} \in X \setminus \{0\}, \langle u^*, \Phi^*_\tau \tilde{\eta} \rangle \geq \sup_{\nu \in L^2([0,\tau], U)} \|\nu\|_{L^2([0,\tau], U)} \leq 1$
4. $\exists \eta \in X \setminus \{0\}, u^* = \Phi^*_\tau \eta$ and $\|u^*\|_{L^2([0,\tau], U)} = 1$

$(A, B)$ controllable implies $\text{Ker} \Phi^*_\tau = \{0\}$. Hence $u^* = \frac{\Phi^*_\tau \tilde{\eta}}{\|\Phi^*_\tau \tilde{\eta}\|_{L^2([0,\tau], U)}}$. 
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If $\tau > T_*$, the sketch of the proof is:

1. $R_t^2 = X$ for every $t > T_*$
2. $z_1 - T_\tau z_0 \in \partial B_T^2(1)$
3. $\exists \tilde{\eta} \in X \setminus \{0\}, \langle u^*, \Phi^{*\tau}_\tau \tilde{\eta} \rangle \geq \sup_{v \in L^2([0,\tau], U)} \langle v, \Phi^{*\tau}_\tau \tilde{\eta} \rangle$
4. $\exists \eta \in X \setminus \{0\}, u^* = \Phi^{*\tau}_\tau \eta$ and $\|u^*\|_{L^2([0,\tau], U)} = 1$
5. $u^*$ is unique
Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of $\tau$ and $u^*$ is ensured by the previous proposition. If $\tau > T_*$, the sketch of the proof is:

1. $R^2_t = X$ for every $t > T_*$
2. $z_1 - T_\tau z_0 \in \partial B^2_\tau(1)$
3. $\exists \tilde{\eta} \in X \setminus \{0\}$, $\langle u^*, \Phi^* \tilde{\eta} \rangle \geq \sup_{v \in L^2([0,\tau], U), \|v\|_{L^2([0,\tau], U)} \leq 1} \langle v, \Phi^* \tilde{\eta} \rangle$
4. $\exists \eta \in X \setminus \{0\}$, $u^* = \Phi^* \eta$ and $\|u^*\|_{L^2([0,\tau], U)} = 1$
5. $u^*$ is unique

If $u_1$ and $u_2$ are time optimal controls, then $\frac{1}{2}(u_1 + u_2)$ is. Hence $\|u_1\| = \|u_2\| = \|\frac{1}{2}(u_1 + u_2)\| = 1$ and the conclusion comes from Cauchy-Schwartz inequality.
Consider the system:

\[ \ddot{y}(x, t) = -\Delta y(x, t) + u(x, t)1_{\mathcal{O}}(x) \quad (x \in \Omega, \quad t \geq 0), \]
\[ y(x, t) = 0 \quad (x \in \partial \Omega, \quad t \geq 0), \]

with initial condition:

\[ (y(\cdot, 0), \dot{y}(\cdot, 0)) = z_0 \in H_0^1(\Omega) \times L^2(\Omega), \]

where \( \Omega \subset \mathbb{R}^n \) is a bonded and open set with \( C^2 \) boundary and \( \mathcal{O} \) is an open subset of \( \Omega \) satisfying the optic geometric condition.

Then for every \( z_1 \in H_0^1(\Omega) \times L^2(\Omega) \), with \( z_1 \neq z_0 \), there exists a time optimal control \( u^* \) steering the solution from \( z_0 \) (at \( t = 0 \)) to \( z_1 \) (at \( t = \tau \)).
Let $T_* \geq 0$ be the time of the optic geometric condition, then if $\tau > T_*$, we have:

- $u^*$ is unique;
- $\|u^*\|_{L^2([0,\tau],U)} = 1$;
- there exists $(\eta_0, \eta_1) \in (H^1_0(\Omega) \times L^2(\Omega)) \setminus \{0\}$ such that
  \[ u^*(x, t) = \dot{w}(x, t) \quad ((x, t) \in \mathcal{O} \times [0, \tau]), \]

where $w$ is solution of the adjoint problem:

\[ \ddot{w}(x, t) = -\Delta w(x, t) \quad (x \in \Omega, \quad t \geq 0), \]
\[ w(x, t) = 0 \quad (x \in \partial\Omega, \quad t \geq 0), \]
\[ w(\cdot, \tau) = \eta_0 \quad \text{and} \quad \dot{w}(\cdot, \tau) = \eta_1. \]
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The finite dimensional problem

Let \((\varphi_n)_{n \in \mathbb{N}}\) be the eigenvectors of \(A\). Since \(A\) is skew-adjoint, \((\varphi_n)_{n}\) can be chosen as an orthonormal basis of \(X\). For every \(N \in \mathbb{N}\), we define \(V_N = \text{Span}\{\varphi_0, \ldots, \varphi_N\}\) and \(P_N \in \mathcal{L}(X)\) the orthogonal projection on \(V_N\). Let also define the problems set for every \(N \in \mathbb{N}\):

**Problem \((\mathcal{P}_N)\)**

Given \(z_0, z_1 \in X\), \(z_0 \neq z_1\) Find the minimal time \(\tau_N \geq 0\) such that there exists a control \(u_N \in L^2([0, \tau_N], U)\) satisfying \(\|u_N\|_{L^2([0,\tau_N],U)} \leq 1\) and \(z_N(\tau_N) = P_N z_1\), where \(z\) is the solution of:

\[
\dot{z}_N = Az_N + P_N Bu_N, \quad z_N(0) = P_N z_0. \tag{*_N}
\]

Notice that the solution of \((*_N)\) is:

\[
z_N(t) = P_N (\mathbb{T}_t z_0 + \Phi_t u_N) \quad (t \in [0, \tau_N]).
\]
Proposition

For every $N \in \mathbb{N}$, $(\mathcal{P}_N)$ admits a unique solution $(\tau_N, u_N)$. In addition, the sequence $(\tau_N)$ is increasing and convergent to $\tau$ and $(E_{\tau_N}^T u_N)_{N \in \mathbb{N}}$ is weakly convergent (up to an extraction) to an element $u \in L^2([0, \tau], U)$ and $(\tau, u)$ is solution of the optimal control problem $(\mathcal{P})$.

In the above, we have defined, for $0 \leq t \leq T$, $E_t^T \in \mathcal{L}(L^2([0, t], U), L^2([0, T], U))$ by:

$$E_t^T v(s) = \begin{cases} v(s) & \text{if } s \in [0, t] , \\ 0 & \text{if } s \in (t, T] , \end{cases} \quad (v \in L^2([0, t], U), \ s \in [0, T]).$$
Strong convergence

**Proposition**

Up to an extraction, the convergence of \((\tau_n, E_{\tau_n}^T u_N)\)_N to a solution \((\tau, u)\) of \((\mathcal{P})\) is strong. In addition,

\[
\|u\|_{L^2([0,\tau], U)} = 1 \quad \text{and} \quad \exists \eta \in X \text{ s.t. } u = \Phi^*_{\tau} \eta.
\]
Strong convergence

Proposition

Up to an extraction, the convergence of \((\tau_n, E^\tau_{\tau_n} u_N))_N\) to a solution \((\tau, u)\) of \((\mathcal{P})\) is strong. In addition,

\[ \| u \|_{L^2([0,\tau], U)} = 1 \quad \text{and} \quad \exists \eta \in X \text{ s.t. } u = \Phi^*_\tau \eta. \]

Main idea of the proof

For every \(N \in \mathbb{N}\), the time optimal control \(u_N\) is the one obtained by the HUM (Hilbert Uniqueness Method), that is:

\[ u_N = \Phi^*_\tau \eta_N, \]

where \(\eta_N \in V_N\) is the minimum on \(V_N\) of the function:

\[ J_N(\zeta) = \frac{1}{2} \int_0^{\tau_N} \| B^* P_N^* \mathbb{T}^*_t \zeta \|^2_U \, dt - \langle P_N(z_1 - \mathbb{T}_{\tau_n} z_0), \zeta \rangle \quad (\zeta \in X). \]

But \((J_N)\) is \(\Gamma\)-convergent to the function \(J\):

\[ J(\zeta) = \frac{1}{2} \int_0^{\tau} \| B^* \mathbb{T}^*_t \zeta \|^2_U \, dt - \langle z_1 - \mathbb{T}_{\tau_n} z_0, \zeta \rangle \quad (\zeta \in X). \]
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4 Conclusion and open questions
- The optimality conditions given still hold for control constraints in $L^p$.
- Results are true for admissible (possibly unbounded) control operators.
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Is the time optimal control unique and of maximal norm for $\tau \leq T_*$?

How can we compute numerically the time optimal control?