Time optimal control for wave-type systems

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Problem Statement

Let X and U be two Banach spaces and $F : X \times U \to X$. Consider the control system:

$$\dot{z} = F(z, u)\,.$$

- Given $\mathrm{z}_0, \mathrm{z}_1 \in X$, $\mathrm{z}_0 \neq \mathrm{z}_1$, show that there exist a minimal time $\tau = \tau(\mathrm{z}_0, \mathrm{z}_1) > 0$ such that there exists a control $u \in L^2([0, \tau], U)$, with

$$\|u\|_{L^2([0,\tau],U)} \leqslant 1\,,$$

for which the solution of the Cauchy problem:

$$\dot{z} = F(z, u)\,, \qquad z(0) = \mathrm{z}_0\,,$$

satisfies:

$$z(T) = \mathrm{z}_1\,;$$

- Derive optimality conditions (Pontryagin's maximum principle);
- Prove the saturation property, i.e. $\|u\|_{L^2([0,\tau],U)} = 1$.

Input to state map

Let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator and let $B \in \mathcal{L}(U, X)$ be a bounded control operator. Then A generate a strongly continuous group of isometries $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$. We consider the dynamical system described by the equation:

$$\dot{z} = Az + Bu\,, \qquad z(0) = \mathrm{z}_0\,, \tag{1}$$

with $u \in L^2([0, T], U)$. Then the solution of (1) writes:

$$z(t) = \mathbb{T}_t \mathrm{z}_0 + \Phi_t u \quad (t \in [0, T])\,,$$

where the maps $\Phi_t \in \mathcal{L}\big(L^2([0, t], U), X\big)$ are the *input to state maps*:

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} Bu(s)\,\mathrm{d}s \qquad (u \in L^2([0, t], U))\,.$$

Controllability notions

Let recall some classical definitions:

Definition

The pair (A, B) *is said:*

- *approximatively controllable in time* τ *if* $\mathrm{Ran}\,\Phi_\tau$ *is dense in* X*;*
- *exactly controllable in time* τ *if* $\mathrm{Ran}\,\Phi_\tau = X$*.*

The L^2 reachable set I

Let $R_t^2 = \Phi_t\big(L^2([0,t],U)\big)$. We endowed R_t^2 with the norm:

$$\|\mathrm{z}\|_{R_t^2} = \inf\{\|u\|_{L^2([0,t],U)}\,,\ \Phi_t u = \mathrm{z}\}\,.$$

Let denote the closed unit ball of R_t^2:

$$B_t^2(1) = \left\{\Phi_t u\,,\ u \in L^2([0,t],U)\,,\ \|u\|_{L^2([0,t],U)} \leqslant 1\right\}\,.$$

The time optimal control problem writes:

Problem $(\mathcal{P})$

Does the set $\left\{T > 0\,,\ \mathrm{z}_1 - \mathbb{T}_T\mathrm{z}_0 \in B_T^2(1)\right\}$ *admits a minimum?*

The L^2 reachable set II

Proposition

Let $0 \leqslant \sigma \leqslant t$. Then we have the continuous inclusions:

$$R^2_\sigma \subset R^2_t \subset X\,.$$

If there exists $T_* \geqslant 0$ such that $R^2_t = X$ for every $t > T_*$, then the norms $\|\cdot\|_X$ and $\|\cdot\|_{R^2_t}$ are equivalent and the *control cost* in time $t > 0$ is defined by:

$$C_t = \sup_{z \neq 0} \frac{\|z\|_{R^2_t}}{\|z\|_X} \qquad (t > T_*)\,.$$

We can see that $t \mapsto C_t$ is non increasing.

Existence of the optimal time

Proposition

Let $\mathrm{z}_0, \mathrm{z}_1 \in X$, $\mathrm{z}_0 \neq \mathrm{z}_1$ *such that there exists* $t \geqslant 0$ *with:*

$$\mathrm{z}_1 - \mathbb{T}_t \mathrm{z}_0 \in B_t^2(1)\,.$$

Then there exist $\tau > 0$ *such that:*

$$\tau = \tau(\mathrm{z}_0, \mathrm{z}_1) = \min\left\{t > 0\,,\; z_1 - \mathbb{T}_t \mathrm{z}_0 \in B_t^2(1)\right\}\,.$$

Optimality conditions

Theorem

Let (A, B) be exactly controllable in any time $T > T_* \geqslant 0$ and $\mathrm{z}_0, \mathrm{z}_1 \in X$, $\mathrm{z}_0 \neq \mathrm{z}_1$, such that there exists $t > 0$ and $u \in L^2([0, t], U)$ with $\|u\|_{L^2([0,t],U)} \leqslant 1$ steering z_0 to z_1 in time t.
Then there exists a minimal time $\tau = \tau(\mathrm{z}_0, \mathrm{z}_1) > 0$ and a time optimal control $u^\star \in L^2([0, \tau], U)$, with $\|u^\star\|_{L^2([0,\tau],U)} \leqslant 1$ steering z_0 to z_1 in time τ.
Moreover, if $\tau > T_*$, then

1. $\exists \eta \in X \setminus \{0\}$ *s.t.* $u^\star = \Phi_\tau^* \eta$;
2. $\|u^\star\|_{L^2([0,\tau],U)} = 1$;
3. $u^\star$ *is unique.*

Idea of the proof

The proof is based on the ideas given by Fattorini. Firstly, the existence of τ and $u^\star$ is ensured by the previous proposition.
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2. $z_1 - \mathbb{T}_\tau z_0 \in \partial B_\tau^2(1)$
    - Assume by contradiction that $z_1 - \mathbb{T}_\tau z_0$ is in the interior of $B_\tau^2(1)$, i.e. there exists $u \in L^2([0,\tau],U)$ such that $\|u\|_{L^2([0,\tau],U)} = r < 1$ and $z_1 - \mathbb{T}_\tau z_0 = \Phi_\tau u$.
    - Take $t \in (T_*, \tau)$ then $z_1 - \mathbb{T}_t z_0 = \Phi_t(u\mathbf{1}_{[0,t]}) + \varphi(\tau,t)$ with $\lim_{t\to\tau} \varphi(\tau,t) = 0$.
    - Since $t \mapsto C_t$ is non increasing, there exists $t \in [T_*,\tau]$ with $\tau - t$ small enough such that there exists $\tilde{u} \in L^2([0,t],U)$, $\|\tilde{u}\| \leqslant 1 - r$ and $\Phi_t \tilde{u} = \varphi(\tau,t)$.
    - Consequently, $u + \tilde{u}$ is a control in time $t < \tau$, with $\|u+\tilde{u}\|_{L^2([0,t],U)} \leqslant 1$.

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    - $B_\tau^2(1)$ is a convex set with non empty interior.
    - $\exists \tilde{\eta} \in X \setminus \{0\}\,,\ \langle \mathrm{z}_1 - \mathbb{T}_\tau \mathrm{z}_0, \tilde{\eta} \rangle \geqslant \sup\limits_{\mathrm{z} \in B_\tau^2(1)} \langle \mathrm{z}, \tilde{\eta} \rangle$

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   (A,B) controllable implies $\mathrm{Ker}\,\Phi_\tau^* = \{0\}$. Hence $u^\star = \dfrac{\Phi_\tau^* \tilde\eta}{\|\Phi_\tau^* \tilde\eta\|_{L^2([0,\tau],U)}}$.

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4. $\exists \eta \in X \setminus \{0\}\,,\ u^\star = \Phi_\tau^* \eta$ and $\|u^\star\|_{L^2([0,\tau],U)} = 1$
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   If u_1 and u_2 are time optimal controls, then $\frac{1}{2}(u_1 + u_2)$ is. Hence $\|u_1\| = \|u_2\| = \|\frac{1}{2}(u_1 + u_2)\| = 1$ and the conclusion comes from Cauchy-Schwartz inequality.

Result for the wave equation I

Consider the system:

$$\begin{aligned} \ddot{y}(x,t) &= -\Delta y(x,t) + u(x,t)\mathbf{1}_{\mathcal{O}}(x) && (x \in \Omega\,, \quad t \geqslant 0)\,, \\ y(x,t) &= 0 && (x \in \partial\Omega\,, \quad t \geqslant 0)\,, \end{aligned}$$

with initial condition:

$$(y(\cdot,0), \dot{y}(\cdot,0)) = \mathrm{z}_0 \in H_0^1(\Omega) \times L^2(\Omega)\,,$$

where $\Omega \subset \mathbb{R}^n$ is a bonded and open set with C^2 boundary and $\mathcal{O}$ is an open subset of Ω satisfying the optic geometric condition.

Then for every $\mathrm{z}_1 \in H_0^1(\Omega) \times L^2(\Omega)$, with $\mathrm{z}_1 \neq \mathrm{z}_0$, there exists a time optimal control $u^\star$ steering the solution from z_0 (at $t = 0$) to z_1 (at $t = \tau$).

Result for the wave equation II

Let $T_* \geqslant 0$ be the time of the optic geometric condition, then if $\tau > T_*$, we have:

- $u^\star$ is unique;
- $\|u^\star\|_{L^2([0,\tau],U)} = 1$;
- there exists $(\eta_0, \eta_1) \in \left(H_0^1(\Omega) \times L^2(\Omega)\right) \setminus \{0\}$ such that

$$u^\star(x,t) = \dot{w}(x,t) \qquad ((x,t) \in \mathcal{O} \times [0,\tau])\,,$$

where w is solution of the adjoint problem:

$$\begin{aligned} \ddot{w}(x,t) &= -\Delta w(x,t) && (x \in \Omega\,, \quad t \geqslant 0)\,, \\ w(x,t) &= 0 && (x \in \partial\Omega\,, \quad t \geqslant 0)\,, \end{aligned}$$

$$w(\cdot,\tau) = \eta_0 \quad \text{and} \quad \dot{w}(\cdot,\tau) = \eta_1\,.$$

The finite dimensional problem

Let $(\varphi_n)_{n\in\mathbb{N}}$ be the eigenvectors of A. Since A is skew-adjoint, $(\varphi_n)_n$ can be chosen as an orthonormal basis of X. For every $N \in \mathbb{N}$, we define $V_N = \mathrm{Span}\{\varphi_0, \dots, \varphi_N\}$ and $P_N \in \mathcal{L}(X)$ the orthogonal projection on V_N. Let also define the problems set for every $N \in \mathbb{N}$:

Problem $(\mathcal{P}_N)$

Given $\mathrm{z}_0, \mathrm{z}_1 \in X$, $\mathrm{z}_0 \neq \mathrm{z}_1$ Find the minimal time $\tau_N \geqslant 0$ such that there exists a control $u_N \in L^2([0, \tau_N], U)$ satisfying $\|u_N\|_{L^2([0,\tau_N],U)} \leqslant 1$ and $z_N(\tau_N) = P_N \mathrm{z}_1$, where z is the solution of:

$$\dot{z}_N = A z_N + P_N B u_N \,, \qquad z_N(0) = P_N \mathrm{z}_0 \,. \tag{$\star_N$}$$

Notice that the solution of $(\star_N)$ is:

$$z_N(t) = P_N(\mathbb{T}_t \mathrm{z}_0 + \Phi_t u_N) \qquad (t \in [0, \tau_N]) \,.$$

Weak convergence

Proposition

For every $N \in \mathbb{N}$, $(\mathcal{P}_N)$ admits a unique solution (τ_N, u_N). In addition, the sequence (τ_N) is increasing and convergent to τ and $(E^{\tau}_{\tau_N} u_N)_{N\in\mathbb{N}}$ is weakly convergent (up to an extraction) to an element $u \in L^2([0,\tau], U)$ and (τ, u) is solution of the optimal control problem $(\mathcal{P})$.

In the above, we have defined, for $0 \leqslant t \leqslant T$,
$E_t^T \in \mathcal{L}\big(L^2([0,t],U), L^2([0,T],U)\big)$ by:

$$E_t^T v(s) = \begin{cases} v(s) & \text{if } s \in [0,t]\,, \\ 0 & \text{if } s \in (t,T]\,, \end{cases} \qquad (v \in L^2([0,t],U)\,, \quad s \in [0,T])\,.$$

Strong convergence

Proposition

Up to an extraction, the convergence of $\left((\tau_n, E^{\tau}_{\tau_n} u_N)\right)_N$ *to a solution* (τ, u) *of* $(\mathcal{P})$ *is strong. In addition,*

$$\|u\|_{L^2([0,\tau],U)} = 1 \quad \text{and} \quad \exists \eta \in X \text{ s.t. } u = \Phi_\tau^* \eta \,.$$

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$$\|u\|_{L^2([0,\tau],U)} = 1 \quad \text{and} \quad \exists \eta \in X \text{ s.t. } u = \Phi^*_\tau \eta\,.$$

Main idea of the proof

For every $N \in \mathbb{N}$, the time optimal control u_N is the one obtained by the HUM (Hilbert Uniqueness Method), that is:

$$u_N = \Phi^*_{\tau_N} \eta_N\,,$$

where $\eta_N \in V_N$ is the minimum on V_N of the function:

$$J_N(\zeta) = \frac{1}{2}\int_0^{\tau_N} \|B^* P_N^* \mathbb{T}_t^* \zeta\|_U^2 \, \mathrm{d}t - \langle P_N(\mathrm{z}_1 - \mathbb{T}_{\tau_n}\mathrm{z}_0), \zeta\rangle \qquad (\zeta \in X)\,.$$

But (J_N) is Γ-convergent to the function J:

$$J(\zeta) = \frac{1}{2}\int_0^{\tau} \|B^* \mathbb{T}_t^* \zeta\|_U^2 \, \mathrm{d}t - \langle \mathrm{z}_1 - \mathbb{T}_{\tau_n}\mathrm{z}_0, \zeta\rangle \qquad (\zeta \in X)\,.$$

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- Results are true for admissible (possibly unbounded) control operators.

- Is the time optimal control unique and of maximal norm for $\tau \leqslant T_*$?
- How can we compute numerically the time optimal control?