A natural splitting method for Kolmogorov-type equations

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Problem Statement

Let us consider the Kolmogorov equation:

$$\partial_t f - x_1 \partial_{x_2} f = \partial_{x_1}^2 f \qquad ((t,x) \in \mathbb{R}_+^* \times \mathbb{R}^2), \qquad (1a)$$

$$f(0,x) = f_0(x) \qquad (x \in \mathbb{R}^2). \qquad (1b)$$

It is easy to see that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^2(\mathbb{R}^2)} = -\|\partial_{x_1} f(t)\|_{L^2(\mathbb{R}^2)}\,.$$

$\longrightarrow$ The diffusion process seems to appear only in the x_1 variable.

But,

$$[\partial_{x_1}, x_1\partial_{x_2}]f = \partial_{x_1}\left(x_1 \partial_{x_2} f\right) - x_1 \partial_{x_2}\partial_{x_1} f = \partial_{x_2} f\,.$$

$\longrightarrow$ There is some hope to recover some diffusion in x_2.

Question

Is it true?

If it is, is it possible to recover this diffusivity by the use of a splitting method?

Let us be more general,

$$\partial_t f + (A(t)x)^\top \nabla f = \rho(t)\partial_{x_1}^2 f\,,$$
$$f(0,x) = f_0(x)\,.$$

Example (Kolmogorov)

$A(t) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ and $\rho(t) = 1$.

The condition $\left[\partial_{x_1},\, (A(t)x)^\top \nabla\right]$ generates a ∂_{x_2} operator implies $[A]_{2,1} \neq 0$. We assume this condition in the following.

Theorem (Young's inequality)

Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and let $f \in L^p(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2)$, then, $f * g \in L^r(\mathbb{R}^2)$ and

$$\|f * g\|_{L^r(\mathbb{R}^2)} \leqslant \|f\|_{L^p(\mathbb{R}^2)} \|g\|_{L^q(\mathbb{R}^2)} .$$

Theorem

Let us assume that $A \in L^\infty(\mathbb{R}_+, M_2(\mathbb{R}))$ *and* $\rho \in L^\infty(\mathbb{R}_+)$, *with* $\rho \geqslant 0$ *and we assume that* $\rho(t) \geqslant \rho_0 > 0$ *and* $[A]_{2,1} \neq 0$ *in a neighborhood of* 0. *Then the solution of*

$$\partial_t f + (A(t)x)^\top \nabla f = \rho(t)\partial_{x_1}^2 f\,, \qquad f(0) = f_0$$

writes:

$$f(t,x) = (f_0 * G_t)\left(\mathcal{A}(t)^{-1}x\right)\,,$$

with:

$$\mathcal{A}(t) = \exp\int_0^t A(s)\,\mathrm{d}s \quad \textit{and} \quad G_t = \frac{1}{4\pi\sqrt{\det\Gamma(t)}}\exp\left(\frac{-1}{4}x^\top\Gamma(t)^{-1}x\right)\,,$$

where,

$$\Gamma(t) = \int_0^t M(s)\,\mathrm{d}s\,, \qquad M(s) = \rho(s)\mathcal{A}(s)^{-1}e_1\left(\mathcal{A}(s)^{-1}e_1\right)^\top\,.$$

With constant coefficients, $A \in M_2(\mathbb{R})$, $\rho = 1$, this writes:

Theorem

The solution of

$$\partial_t f + (Ax)^\top \nabla f = \partial_{x_1}^2 f \,, \qquad f(0) = f_0$$

writes:

$$f(t,x) = (f_0 * G_t)\,(\mathcal{A}(-t)x) \ ,$$

with:

$$\mathcal{A}(t) = \exp(tA) \quad \textit{and} \quad G_t = \frac{1}{4\pi\sqrt{\det\Gamma(t)}} \exp\left(\frac{-1}{4} x^\top \Gamma(t)^{-1} x\right) \ ,$$

where,

$$\Gamma(t) = \int_0^t M(s)\,\mathrm{d}s \,, \qquad M(s) = \mathcal{A}(-s)e_1\,(\mathcal{A}(-s)e_1)^\top \ .$$

Steps of the proof.

1. Solve the convection equation;

$$\frac{\mathrm{d}\,X}{\mathrm{d}t} = AX\,, \quad X(0) = x \qquad \longrightarrow \qquad X = \exp(tA)x\,.$$

2. Change variables;

$$g(t,x) = f(t,X) \qquad \longrightarrow \quad \partial_t g = \mathrm{div}\left(M(t)\nabla g\right)\,, \quad g(0) = f(0)\,.$$

3. Fourier transformation;

$$\hat{g}(t,\xi) = \int_{\mathbb{R}^2} g(t,x) e^{-ix^\top \xi}\,\mathrm{d}x \qquad \longrightarrow \qquad \partial_t \hat{g} = -\xi^\top M(t)\xi\hat{g}$$

$$\hat{g}(t,\xi) = \exp\left(-\xi^\top \int_0^t M(s)\,\mathrm{d}s\;\xi\right)\hat{f}_0(\xi) = \exp\left(-\xi^\top \Gamma(t)\xi\right)\hat{f}_0(\xi)\,.$$

4. Inverse Fourier transformation;
   Using Cauchy-Schwartz and $[A]_{2,1} \neq 0$, we have $\det \Gamma(t) > 0$ and in addition $\mathrm{Tr}\,\Gamma(t) > 0$.

Using Young's Inequality, we obtain:

Corollary (Decay rate)

Let $p, q \in [1, \infty]$, *if* $f_0 \in L^p(\mathbb{R}^2)$ *then for every* $t > 0$, *we have* $f(t) \in L^r(\mathbb{R}^2)$ *with* $r \in [1, \infty]$ *such that* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and*

$$\|f(t)\|_{L^r(\mathbb{R}^2)} \leqslant |\det \mathcal{A}(t)|^{\frac{1}{r}} \left(4\pi\sqrt{\det \Gamma(t)}\right)^{\frac{1-q}{q}} \|f_0\|_{L^p(\mathbb{R}^2)} \qquad (t > 0).$$

Example (Kolmogorov)

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{A}(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}, \qquad \Gamma(t) = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix}.$$

$$f(t, x_1, x_2) = (f_0 * G_t)(x_1, x_2 + tx_1),$$

$$\text{with } G_t(x_1, x_2) = \frac{\sqrt{3}}{2\pi t^2} \exp\left(\frac{-3}{t^3}(t^2x_1^2 - 3tx_1x_2 + 3x_2^2)\right).$$

$$\text{In addition, } \|f(t)\|_{L^r(\mathbb{R}^2)} \leqslant \left(\frac{2\pi t^2}{\sqrt{3}}\right)^{\frac{1-q}{q}} q^{\frac{-1}{q}} \|f_0\|_{L^p(\mathbb{R}^2)} \qquad (t > 0).$$

Let $\tau > 0$, we consider the scheme:

- Initialisation;

$$f_0 = f_0.$$

- Iterative procedure;

1.
$$\begin{aligned} \partial_t h_1 &= -(Ax)^\top \nabla h_1\,, \\ h_1(0) &= f_n\,, \end{aligned}$$

2.
$$\begin{aligned} \partial_t h_2 &= \partial_{x_1}^2 h_2\,, \\ h_2(0) &= h_1(\tau)\,. \end{aligned}$$

3.
$$f_{n+1} = h_2(\tau).$$

Let χ be the indicator of $\bigcup_{n\in\mathbb{N}} \left[n\tau, \left(n+\frac{1}{2}\right)\tau\right)$, then,

$$f_n = h(n\tau) \qquad (n \in \mathbb{N})\,,$$

with,

$$\begin{aligned} \partial_t h &= 2\chi(t)\partial_{x_1}^2 h + 2(1-\chi(t))(Ax)^\top \nabla h\,, \\ h(0) &= f_0\,. \end{aligned}$$

Using the first Theorem, we obtain,

Corollary

For $\tau > 0$ small enough and every $n \in \{1, \dots, N(\tau)\}$, we have:

$$f_n(x) = (f_0 * G^n)\left(\mathcal{A}(-n\tau)x\right) \qquad (x \in \mathbb{R}^2),$$

with:

$$\mathcal{A}(t) = \exp(tA) \quad \textit{and} \quad G^n = \frac{1}{4\pi\sqrt{\det \Gamma_n}} \exp\left(\frac{-1}{4} x^\top \Gamma_n^{-1} x\right),$$

where,

$$\Gamma_n = \sum_{k=0}^{n-1} \tau M(k\tau), \qquad M(s) = \mathcal{A}(-s)e_1 \left(\mathcal{A}(-s)e_1\right)^\top .$$

In addition, if $f_0 \in L^p(\mathbb{R}^2)$, we have:

$$\|f_n\|_{L^r(\mathbb{R}^2)} \leqslant |\det \mathcal{A}(n\tau)|^{\frac{1}{r}} \left(4\pi\sqrt{\det \Gamma_n}\right)^{\frac{1-q}{q}} \|f_0\|_{L^p(\mathbb{R}^2)} .$$

with $p, q, r \in [1, \infty]$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Remark

$\Gamma_n = \sum_{k=0}^{n-1} \tau M(k\tau)$ *is the Riemann approximation of* $\Gamma(n\tau) = \int_0^{n\tau} M(s)\,\mathrm{d}s$.

Let $R_n = \Gamma(n\tau) - \Gamma_n = \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} \int_{k\tau}^{s} M'(\sigma)\,\mathrm{d}\sigma$, *we have:*

$$\|M(t)\|_{M_2(\mathbb{R})} \leqslant \exp\left(2t\|A\|_{M_2(\mathbb{R})}\right) \quad \textit{and} \quad M'(t) = -\left(AM(t) + M(t)A^\top\right),$$

Hence,

$$\|R_n\|_{M_2(\mathbb{R})} \leqslant \epsilon(\tau) \exp\left(2n\tau\|A\|_{M_2(\mathbb{R})}\right),$$

with $\epsilon(\tau) = \mathrm{O}_0(\tau)$.

$\longrightarrow$ *Is it possible to have valid long time simulations with this method?*

Let us focus on the Kolmogorov equation:

$$\partial_t f - x_1 \partial_{x_2} f = \partial_{x_1}^2 f \qquad ((t,x) \in \mathbb{R}_+^* \times \mathbb{R}^2).$$

We remind that:

$$f(n\tau, x) = (f_0 * G_{n\tau})(\mathcal{A}(-n\tau)x)\,,$$
$$f_n(x) = (f_0 * G^n)(\mathcal{A}(-n\tau)x)\,,$$

with:

$$\mathcal{A}(t) = \exp(tA) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \text{ and } \begin{cases} G_t & = & \dfrac{1}{4\pi\sqrt{\det \Gamma(t)}} \exp\left(\dfrac{-1}{4} x^\top \Gamma(t)^{-1} x\right) \\ G^n & = & \dfrac{1}{4\pi\sqrt{\det \Gamma_n}} \exp\left(\dfrac{-1}{4} x^\top \Gamma_n^{-1} x\right), \end{cases}$$

where,

$$\Gamma(t) = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix} \text{ and } \Gamma_n = \Gamma(n\tau) - \frac{n\tau^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & n\tau - \frac{\tau}{3} \end{pmatrix}.$$

Proposition

Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and assume that $f_0 \in L^p(\mathbb{R}^2)$, then there exists a constant $C_n \geqslant 0$ (depending only on n, τ, q) such that:

$$\|f(n\tau) - f_n\|_{L^r(\mathbb{R}^2)} \leqslant C_n \|G_{n\tau}\|_{L^q(\mathbb{R}^2)} \|f_0\|_{L^p(\mathbb{R}^2)} ,$$

where,

$$\lim_{n \to \infty} C_n = 0 .$$

Remark

$$\|f(t)\|_{L^r(\mathbb{R}^2)} \leqslant \|G_t\|_{L^q(\mathbb{R}^2)} \|f_0\|_{L^p(\mathbb{R}^2)} ,$$

Proof (started).

It is easy to see:

$$\|f(n\tau) - f_n\|_{L^r(\mathbb{R}^2)} = \|f_0 * (G_{n\tau} - G^n)\|_{L^r(\mathbb{R}^2)} \leqslant \|f_0\|_{L^p(\mathbb{R}^2)} \|G_{n\tau} - G^n\|_{L^q(\mathbb{R}^2)} .$$

Then we only need to prove

$$\|G_{n\tau} - G^n\|_{L^q(\mathbb{R}^2)} \leqslant C_n \|G_{n\tau}\|_{L^q(\mathbb{R}^2)} .$$

□

Proof (continued).

Let $\varphi_n = \dfrac{G^n}{G_{n\tau}}$. Then $G_{n\tau} - G^n = (1-\varphi_n)G_{n\tau}$.

$\longrightarrow$ First difficulty, φ_n is not bounded on $\mathbb{R}^2$

Idea: Use fact that φ_n is close to one around 0 and eat the growth of φ_n at infinity by the decay of $G_{n\tau}$.

That is to say, given $\varepsilon \in (0,1)$ and $R > 0$, we write:

$$G_{n\tau} - G^n = (1-\varphi_n)\mathbf{1}_{B(R)}\, G_{n\tau} + (1-\varphi_n)G_{n\tau}^{\varepsilon}\, G_{n\tau}^{1-\varepsilon}\mathbf{1}_{\mathbb{R}^2\setminus B(R)}\,.$$

Hence,

$$\begin{aligned}\|G_{n\tau} - G^n\|_{L^q(\mathbb{R}^2)} \leqslant \|1-\varphi_n\|_{L^\infty(B(R))}\|G_{n\tau}\|_{L^q(\mathbb{R}^2)} \\ + \|(1-\varphi_n)G_{n\tau}^{\varepsilon}\|_{L^\infty(\mathbb{R}^2)}\|G_{n\tau}^{1-\varepsilon}\|_{L^q(\mathbb{R}^2\setminus B(R))}\,.\end{aligned}$$

□

Proof (finished).

One can chose $\varepsilon = \varepsilon_n$ such that $\sup_{n\in\mathbb{N}} \|(1-\varphi_n)G_{n\tau}^{\varepsilon_n}\|_{L^\infty(\mathbb{R}^2)} < \infty$ and

$$\|G_{n\tau}^{1-\varepsilon_n}\|_{L^q(\mathbb{R}^2\setminus B(R))} \leqslant P(n)e^{-\mu_n R^2}\|G_{n\tau}\|_{L^q(\mathbb{R}^2)}.$$

In addition, we have

$$\|1-\varphi_n\|_{L^\infty(B(R))} \leqslant e^{\lambda_n R^2} - 1$$

and λ_n and μ_n are such that it is possible to find $R = R_n$ such that

$$\lim_{n\to\infty} \lambda_n R_n^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \mu_n R_n^2 = +\infty,$$

That is to say,

$$\lim_{n\to\infty} e^{\lambda_n R_n^2} - 1 + P(n)e^{-\mu_n R_n^2} = 0.$$

□

Open Questions

- Is the study true for every matrix A?

- Numerical methods?

- What about:

$$\partial_t f - x_1 \partial_{x_2} f = \partial_{x_1} \left(\sigma(x_1) \partial_{x_1} f \right) \ ?$$

- Fokker-Planck,

$$\partial_t f + W(x)^\top \nabla f = \partial_{x_1}^2 f \,,$$

with $W(x) = \begin{pmatrix} x_1 - \alpha(x_2) \\ x_1 \end{pmatrix}$?