Stationary phase lemma and its applications

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RT1-PDEs Working Group activity
Motivation: wave equations with oscillatory initial data
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Stationary phase: non-singular phase
Outline

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Motivation: The wave equation with oscillatory initial data

Consider the Cauchy problem for the wave equation with an oscillatory initial data:

\[
\begin{align*}
&u_{tt}^\epsilon(x, t) - \Delta u^\epsilon(x, t) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
u^\epsilon(x, 0) = g^\epsilon(x), \quad u_t^\epsilon(x, 0) = u^{\epsilon,1}(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

with \(\widehat{g}^\epsilon(x) = a(x) \exp\left(\frac{ip(x)}{\epsilon}\right)\) and \(\widehat{u}^{\epsilon,1}(\xi) = i|\xi|\widehat{g}^\epsilon(\xi), \quad a, p \in C^\infty_c(\mathbb{R}^d)\).

The solution of the problem (1) is given by

\[
u^\epsilon(x, t) = \frac{1}{(2\pi \epsilon)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(y) \exp\left(\frac{i\phi(x, t, y, \eta)}{\epsilon}\right) \, dy \, d\eta,
\]

with \(\phi(x, t, y, \eta) = p(y) + \eta \cdot (x - y) + t|\eta|\).

How does \(u^\epsilon(x, t)\) behave as \(\epsilon \to 0\)?

**Answer:** Depends on the singular points of \(\phi\) in the variables \((y, \eta)\).

**Aim:** the asymptotic behavior as \(\epsilon \to 0\) of the integral

\[
I_\epsilon := \int_{\mathbb{R}^d} a(x) \exp\left(\frac{i\phi(x)}{\epsilon}\right) \, dx,
\]

where \(a \in C^\infty_c(\mathbb{R}^d)\) (amplitude) and \(\phi\) (phase) is some "smooth" function.
Getting intuition using Taylor approximations of integrands

\[ I_\epsilon = \int_{x^-}^{x^+} a(x) \exp(i\phi(x)/\epsilon) \, dx. \]

First approximation of the integrand

\[ a(x_0) \exp \left( i \left( \phi(x_0) + (x - x_0)\phi'(x_0) \right)/\epsilon \right), \]

whose primitive is

\[ \frac{\epsilon}{i\phi'(x_0)} a(x_0) \exp \left( i \left( \phi(x_0) + (x - x_0)\phi'(x_0) \right)/\epsilon \right), \]

provided \( \phi'(x_0) \neq 0. \)

If \( x_0 = x^\pm \), the value of the primitive at \( x_0 \) is \( \pm \frac{\epsilon \exp(i\phi(x_0))a(x_0)}{i\phi'(x_0)} \)

If \( \phi'(x_0) = 0 \), an approximation of the integrand is

\[ a(x_0) \exp \left( i \left( \phi(x_0) + (x - x_0)^2\phi''(x_0)/2 \right)/\epsilon \right). \]

\[ I_\epsilon \sim \exp(\pm \pi i/4) a(x_0) \exp(i\phi(x_0)/\epsilon) \left| \frac{2\pi\epsilon}{\phi''(x_0)} \right|^{1/2}. \]
Lemma (linear case, see [1], p. 208-217)

Let $a \in C_c^\infty(\mathbb{R}^d)$, $p \in \mathbb{R}^d$, $p \neq 0$. Then, for all $m \in \mathbb{N}^*$,

$$I_\epsilon = \int_{\mathbb{R}^d} a(x) \exp \left( \frac{ip \cdot x}{\epsilon} \right) \, dx = O(\epsilon^m) \text{ as } \epsilon \to 0.$$

Sketch of proof. Assume $p_1 \neq 0$ and integrate $m$ times by parts in $x_1$:

$$I_\epsilon = \int_{\mathbb{R}^d} \left( \frac{\epsilon}{ip_1} \right)^m \frac{\partial^m}{\partial x_1^m} \left( \exp \left( \frac{ip \cdot x}{\epsilon} \right) \right) a(x) \, dx = \left( - \frac{\epsilon}{ip_1} \right)^m \int_{\mathbb{R}^d} \exp \left( \frac{ip \cdot x}{\epsilon} \right) \frac{\partial^m a(x)}{\partial x_1^m} \, dx$$

$$\leq \left( \frac{\epsilon}{p_1} \right)^m \| \partial_{x_1}^m a \|_{L^1(\mathbb{R}^d)}.$$

Lemma (non-singular phase, $d = 1$, see [2], p. 329-432)

Let $a \in C_c^\infty(a, b)$, $\phi$ be smooth function s.t. $\phi'(x) \neq 0 \ \forall x \in [a, b]$. Then

$$I_\epsilon = O(\epsilon^m), \ \forall m \in \mathbb{N}.$$
Sketch of proof. Set \((\mathcal{L}a)(x) = (a(x)/i\phi'(x))'\). Then

\[
I_\epsilon = (-\epsilon)^m \int_{\mathbb{R}} \exp \left( \frac{i\phi(x)}{\epsilon} \right) (\mathcal{L}^m a)(x) \, dx.
\]

**Lemma (non-singular phase, see [1], vol. I, p. 215)**

Let \(K \subset \mathbb{R}^d\) compact. If \(a \in C^k_c(K), \phi \in C^{k+1}(K)\) and \(\text{Im}(\phi) \geq 0\), then

\[
\epsilon^{-(k+j)} \left| \int_{\mathbb{R}^d} a(x) \left( \text{Im}(\phi(x)) \right)^j \exp \left( \frac{i\phi(x)}{\epsilon} \right) \, dx \right|
\]

\[
\leq C(j) \sum_{|\alpha| \leq k} \sup_{x \in K} |D^\alpha a(x)| \left( |\nabla \phi(x)|^2 + \text{Im}(\phi(x)) \right)^{\frac{|\alpha|}{2} - k}. \tag{3}
\]

Lemma (quadratic case, asymptotic expansion)

Let $a \in C_c^\infty (\mathbb{R}^d)$ and $A$ - real, non-singular, symmetric matrix. Then

\[
\left| \int_{\mathbb{R}^d} a(x) \exp \left( \frac{i}{\varepsilon} \frac{1}{2} (Ax, x) \right) \, dx - \frac{(2\pi i \varepsilon)^{d/2}}{(\det(A))^{1/2}} T_k(\varepsilon) \right| \\
\leq C_k (\varepsilon \| A^{-1} \|)^{d/2 + k} \sum_{|\alpha| \leq 2k + s} \| D^\alpha a \|_{L^2(\mathbb{R}^d)},
\]

with $T_k(\varepsilon) = \sum_{j=0}^{k-1} \left( \frac{\varepsilon}{2i} \right)^j (A^{-1} D, D)^j \frac{a(0)}{j!}$.

Sketch of proof. Define $\alpha_\varepsilon(x) := \exp \left( \frac{i}{\varepsilon} \frac{1}{2} (Ax, x) \right)$.

By Parseval identity,

\[
\int_{\mathbb{R}^d} a(x) \alpha_\varepsilon(x) \, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{a}(\xi) \hat{\alpha}_\varepsilon(-\xi) \, d\xi.
\]

\[
\hat{\alpha}_\varepsilon(\xi) = \frac{(2\pi i \varepsilon)^{d/2}}{|\det(A)|^{1/2}} \exp \left( \frac{\pi i}{4} \text{sgn}(A) \right) \exp \left( - \frac{i\varepsilon}{2} (A^{-1} \xi, \xi) \right).
\]
Taylor expansion:

\[
\exp \left( -\frac{i\epsilon}{2} (A^{-1} \xi, \xi) \right) = \sum_{j=0}^{k-1} \frac{1}{j!} \left( -\frac{i\epsilon}{2} (A^{-1} \xi, \xi) \right)^j + \frac{1}{k!} \left( -\frac{i\epsilon}{2} (A^{-1} \xi, \xi) \right)^k \exp(i\theta).
\]

Use the identity:

\[
\sum_{j=0}^{k-1} \frac{1}{j!} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{a}(\xi) \left( \frac{i\epsilon}{2} (A^{-1} i\xi, i\xi) \right)^j d\xi = \sum_{j=0}^{k-1} \frac{1}{j!} \left( \frac{i\epsilon}{2} \right)^j ((A^{-1} \nabla, \nabla)^j a)(0) = T_k(\epsilon).
\]

The error term:

\[
\text{Error} \leq \frac{(2\pi\epsilon)^{d/2}}{|\det(A)|^{1/2}} \frac{1}{k!} \left( \frac{\epsilon\|A^{-1}\|}{2} \right)^k \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{a}(\xi)||\xi|^{2k} d\xi.
\]

\[
\int_{\mathbb{R}^d} |\hat{a}(\xi)||\xi|^{2k} d\xi \leq 2\pi^{d/2} \omega_d^{1/2} \|a\|_{L^2(\mathbb{R}^d)} + 2\pi^{d/2} \omega_d^{1/2} \frac{1}{\sqrt{2s - d}} \|D^{2k+s} a\|_{L^2(\mathbb{R}^d)}.
\]
**General phase**

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**Lemma**

Assume $\phi : \mathbb{R}^d \to \mathbb{R}$ is smooth.

i. Suppose $\nabla \phi(0) \neq 0$. Then there exists a smooth function $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ s.t.

\[
\begin{cases}
\overrightarrow{\varphi}(0) = 0, \quad D \overrightarrow{\varphi}(0) = I \\
\phi(\overrightarrow{\varphi}(x)) = \phi(0) + \nabla \phi(0) \cdot x, \quad |x| \text{ small}.
\end{cases}
\]

ii. (Morse Lemma) Suppose that $\nabla \phi(0) = 0$, $\det(D^2 \phi(0)) \neq 0$. Then there exists a smooth function $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ s.t.

\[
\begin{cases}
\overrightarrow{\varphi}(0) = 0, \quad D \overrightarrow{\varphi}(0) = I \\
\phi(\overrightarrow{\varphi}(x)) = \phi(0) + \frac{1}{2} x \cdot D^2 \phi(0)x, \quad |x| \text{ small}.
\end{cases}
\]

**Sketch of proof.** i. Set $v_d = D\phi(0) \neq 0$ and add $v_1, \cdots, v_{d-1}$ s.t. $(v_i)_{i=1, \cdots, d}$ orthogonal basis in $\mathbb{R}^d$.

Apply Implicit Functions Theorem to

\[
f(x, y) = (v_1 \cdot (x - y), \cdots, v_{d-1} \cdot (x - y), \phi(y) - \phi(0) - v_d \cdot x).
\]

Since $D_y f(0, 0) \neq 0$, there exists $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $\overrightarrow{\varphi}(0) = 0$ and $f(x, \overrightarrow{\varphi}(x)) = 0$ for small $|x|$.

In particular, $\phi(\overrightarrow{\varphi}(x)) = \phi(0) + v_d \cdot x$, $(\overrightarrow{\varphi}(x) - x) \cdot v_i = 0$, $i = 1, \cdots, d-1$. 

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By differentiation in $x$, $(D \varphi(0) - I) \cdot v_i = 0$, $i = 1, \cdots, d$. Then $D \varphi(0) = I$.

ii. Since $D\phi(0) = 0$, $\phi(x) = \phi(0) + \frac{1}{2} (A(x)x, x)$, with $A(x) = 2 \int_0^1 (1 - t) D^2 \phi(tx) \, dt$, with $A(0) = D^2 \phi(0)$.

$A(0)$-non-singular. For small $|x|$, $A(x)$-non-singular.

By a rotation, assume $A(0)$-diagonal.

By induction, for each $m = 0, \cdots, d$, construct $\varphi_m : \mathbb{R}^d \rightarrow \mathbb{R}^d$ s.t.

\[
\begin{cases}
\varphi_m(0) = 0, & D \varphi_m(0) = I \\
\phi(\varphi_m(x)) = \phi(0) + \frac{1}{2} \sum_{i=1}^m \phi_{x_i,x_i}(0)x_i^2 + \frac{1}{2} \sum_{i,j=m+1}^d a_{ij}^m(x)x_ix_j.
\end{cases}
\]

This means $a_{ij}^m(0) = \phi_{x_i,x_j}(0)$. In particular $a_{m+1,m+1}^m(x) \neq 0$ for small $|x|$.

For $m = 0$, $A_0(x) = A(x)$ and $\varphi_0(x) = I$.

Assume $\varphi_m$ and $A_m$ defined. Define the mapping $\Pi^{m+1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\Pi^{m+1}(y) = x$ by $\Pi^{m+1}(y) =$

\[
\left( y_1, \cdots, y_m, \left( \frac{a_{m+1,m+1}^m(y)}{\phi_{x_{m+1},x_{m+1}}(0)} \right)^{1/2} \left( y_{m+1} + \sum_{j=m+2}^d \frac{a_{m+1,j}^m(y)}{a_{m+1,m+1}^m(y)} y_j \right), y_{m+2}, \cdots, y_d \right).
\]
Then
\[
\phi(\varphi_m(y)) = \phi(0) + \frac{1}{2} \sum_{j=1}^{m+1} \phi_{x_j, x_j}(0) x_j^2 + \frac{1}{2} \sum_{i, j = m+2}^{d} b_{ij}^{m+1}(y)x_i x_j,
\]

\[
b_{ij}^{m+1}(y) = \begin{cases} 
    a_{ij}^m(y) - \frac{a_{m+1, i}^m(y)a_{m+1, j}^m(y)}{a_{m+1, m+1}^m(y)}, & i, j = m + 2, \cdots, d \\
    0, & \text{otherwise.}
\end{cases}
\]

Since $D^2 \phi(0)$ -diagonal, $\Pi^{m+1}(0) = 0$, $D \Pi^{m+1}(0) = I$.

Then, $(\Pi^{m+1})^{-1}$ well defined.

Set $A^{m+1} = B^{m+1} \circ (\Pi^{m+1})^{-1}$,

$\varphi_{m+1} = \varphi_m \circ (\Pi^{m+1})^{-1}$. 
The stationary phase method

Partition of unity argument:
Suppose $\nabla \phi$ vanishes within the support of $a$ only at a finite number of points, $x_1, \cdots, x_N$ and that $D^2 \phi(x_j)$ is non-singular, $\forall j = 1, \cdots, N$.
Consider $\varsigma \in C_c^\infty(\mathbb{R}^d)$ which vanishes near each $x_i$. Then

$$\left| \int_{\mathbb{R}^d} a(x) \varsigma(x) \exp \left( \frac{i\phi(x)}{\epsilon} \right) \, dx \right| = O(\epsilon^m), \quad \forall m \in \mathbb{N}.$$  

$$\left| \int_{\mathbb{R}^d} a(x) (1 - \varsigma(x)) \exp \left( \frac{i\phi(x)}{\epsilon} \right) \, dx \right| - \sum_{j=1}^{N} \exp(i\phi(x_k)/\epsilon) \frac{(2\pi\epsilon)^{d/2}}{|\det(D^2 \phi(x_j))|^{1/2}} \exp \left( \frac{i\pi}{4} \text{sign}(D^2 \phi(x_j)) \right) a(x_j) \right| = O(\epsilon^{d/2+1}).$$  

(7)

Lemma (higher order singularities, van der Corput)

Suppose $\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0$ and $\phi^{(k)}(x_0) \neq 0$. Then, if $a$ is supported in a sufficiently small neighborhood of $x_0$, $I_\epsilon \sim \epsilon^{1/k} \sum_{j=0}^{\infty} a_j \epsilon^{j/k}$.
The behavior of the Bessel functions at infinity

\[ J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ir \sin(\theta)) \exp(-im\theta) d\theta. \]

\( \epsilon = 1/r, \phi(\theta) = \sin(\theta). \) Singular points: \( \theta_1 = \pi/2, \theta_2 = 3\pi/2 \) and \( \phi''(\theta_i) = \pm1. \)

\[ J_m(r) = O(r^{-1/2}), \text{ as } r \to \infty. \]

Riemann singularity

\[ \int_0^1 \exp(i\xi x) \exp(i/x) x^{-\gamma} dx = \sqrt{\pi i} \exp(2i\xi^{1/2}) \xi^{-3/4+\gamma/2} + O(\xi^{-1+\gamma/2}), \]

as \( \xi \to \infty, 0 \leq \gamma < 2. \)
\( \phi(x) = x^{-1} + x\xi, \) whose singular point is \( x_0 = \xi^{-1/2}, \phi''(x_0) = 2\xi^{3/2} \)

Airy function

\[ Ai(-x) = \frac{1}{\pi} \int_0^\infty \exp \left( i \left( \frac{1}{3} \xi^3 - x\xi \right) \right) d\xi = \pi^{-1/2} x^{-1/4} \exp \left( i \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) \right) + o(x^{-1/4}). \]

\( \phi(\xi) = \xi^3/3 - \xi x, \) whose critical point is \( \xi_0 = x^{1/2}. \)
Stationary phase method applied to the wave equation (1)

\[ \phi(x, t, y, \eta) = p(y) + \eta \cdot (x - y) + t|\eta|. \]

The set of stationary points \( S = \{(y, \eta) : x = y - t \frac{\nabla p(y)}{|\nabla p(y)|}, \eta = \nabla p(y)\} \).

The Hessian matrix

\[ D^2_{y,\eta} \phi = \begin{pmatrix} D^2 p(y) & -I \\ -I & \frac{t}{|\eta|} (I - \frac{\eta \otimes \eta}{|\eta|^2}) \end{pmatrix}. \]

\[ \det(D^2_{y,\eta} \phi) = (-1)^d \det \left( I - \frac{t}{|\nabla p|} D^2 p(I - \frac{\nabla p \otimes \nabla p}{|\nabla p|^2}) \right) = (-1)^d \prod_{i=1}^{d-1} (1 - tk_i(y)), \]

\( k_i(y) \)- the principal curvatures.

For \( t \) sufficiently small, use Implicit Functions Theorem to find \( y_0 = y_0(x, t), \eta_0 = \eta_0(x, t) \) the singular points.