Stationary phase lemma and its applications

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RT1-PDEs Working Group activity

- Motivation: wave equations with oscillatory initial data

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- Stationary phase: non-singular phase

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- Stationary phase: non-singular phase
- Stationary phase: quadratic case
- Stationary phase: general phase
- Applications
- The wave equation and connection to Geometric Optics

Consider the Cauchy problem for the wave equation with an oscillatory initial data:

$$\begin{cases} u^\epsilon_{tt}(x,t) - \Delta u^\epsilon(x,t) = 0, & x \in \mathbb{R}^d, t > 0 \\ u^\epsilon(x,0) = g^\epsilon(x), u^\epsilon_t(x,0) = u^{\epsilon,1}(x), & x \in \mathbb{R}^d, \end{cases} \quad (1)$$

with $\widehat{g}^\epsilon(x) = a(x)\exp\left(\frac{ip(x)}{\epsilon}\right)$ and $\widehat{u}^{\epsilon,1}(\xi) = i|\xi|\widehat{g}^\epsilon(\xi)$, $a, p \in C^\infty_c(\mathbb{R}^d)$.

The solution of the problem (1) is given by

$$\boxed{u^\epsilon(x,t) = \frac{1}{(2\pi\epsilon)^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(y)\exp\left(\frac{i\phi(x,t,y,\eta)}{\epsilon}\right) dy\, d\eta,} \quad (2)$$

with $\phi(x,t,y,\eta) = p(y) + \eta\cdot(x-y) + t|\eta|$.

How does $u^\epsilon(x,t)$ behave as $\epsilon \to 0$?

Answer: Depends on the singular points of ϕ in the variables (y,η).

Aim: **the asymptotic behavior as** $\epsilon \to 0$ of the integral

$$\boxed{I_\epsilon := \int\limits_{\mathbb{R}^d} a(x)\exp\left(\frac{i\phi(x)}{\epsilon}\right) dx,}$$

where $a \in C^\infty_c(\mathbb{R}^d)$ (amplitude) and ϕ (phase) is some "smooth" function.

$$I_\epsilon = \int_{x^-}^{x^+} a(x) \exp(i\phi(x)/\epsilon)\, dx.$$

First approximation of the integrand

$$a(x_0) \exp\Big(i\big(\phi(x_0) + (x - x_0)\phi'(x_0)\big)/\epsilon\Big),$$

whose primitive is

$$\frac{\epsilon}{i\phi'(x_0)} a(x_0) \exp\Big(i\big(\phi(x_0) + (x - x_0)\phi'(x_0)\big)/\epsilon\Big),$$

provided $\phi'(x_0) \neq 0$.
If $x_0 = x^\pm$, the value of the primitive at x_0 is $\pm\frac{\epsilon \exp(i\phi(x_0))a(x_0)}{i\phi'(x_0)}$
If $\phi'(x_0) = 0$, an approximation of the integrand is

$$a(x_0) \exp\Big(i\big(\phi(x_0) + (x - x_0)^2\phi''(x_0)/2\big)/\epsilon\Big).$$

$$I_\epsilon \sim \exp(\pm\pi i/4) a(x_0) \exp(i\phi(x_0)/\epsilon) \left|\frac{2\pi\epsilon}{\phi''(x_0)}\right|^{1/2}.$$

Lemma (linear case, see [1], p. 208-217)

Let $a \in C_c^\infty(\mathbb{R}^d)$, $p \in \mathbb{R}^d$, $p \neq 0$. *Then, for all* $m \in \mathbb{N}^*$,

$$I_\epsilon = \int\limits_{\mathbb{R}^d} a(x) \exp\left(\frac{ip \cdot x}{\epsilon}\right) dx = O(\epsilon^m) \text{ as } \epsilon \to 0.$$

Sketch of proof. Assume $p_1 \neq 0$ and integrate m times by parts in x_1:

$$I_\epsilon = \int\limits_{\mathbb{R}^d} \left(\frac{\epsilon}{ip_1}\right)^m \frac{\partial^m}{\partial x_1^m}\left(\exp\left(\frac{ip \cdot x}{\epsilon}\right)\right) a(x)\, dx = \left(-\frac{\epsilon}{ip_1}\right)^m \int\limits_{\mathbb{R}^d} \exp\left(\frac{ip \cdot x}{\epsilon}\right) \frac{\partial^m a(x)}{\partial x_1^m}\, dx$$

$$\leq \left(\frac{\epsilon}{p_1}\right)^m \|\partial_{x_1}^m a\|_{L^1(\mathbb{R}^d)}.$$

Lemma (non-singular phase, $d = 1$,see [2], p. 329-432)

Let $a \in C_c^\infty(a,b)$, ϕ *be smooth function s.t.* $\phi'(x) \neq 0\ \forall x \in [a,b]$. *Then*

$$I_\epsilon = O(\epsilon^m),\ \forall m \in \mathbb{N}.$$

L. Evans, *Partial Differetial Equations*, AMS, 1998

E. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, 1993.

Sketch of proof. Set $(\mathcal{L}a)(x) = \big(a(x)/i\phi'(x)\big)'$. Then

$$I_\epsilon = (-\epsilon)^m \int_{\mathbb{R}} \exp\left(\frac{i\phi(x)}{\epsilon}\right)(\mathcal{L}^m a)(x)\,dx.$$

Lemma (non-singular phase, see [1], vol. I, p. 215)

$K \subset \mathbb{R}^d$ compact. If $a \in C_c^k(K)$, $\phi \in C^{k+1}(K)$ and $Im(\phi) \geq 0$, then

$$\epsilon^{-(k+j)}\left|\int_{\mathbb{R}^d} a(x)\Big(Im(\phi(x))\Big)^j \exp\left(\frac{i\phi(x)}{\epsilon}\right)dx\right|$$

$$\leq C(j)\sum_{|\alpha|\leq k}\sup_{x\in K}|D^\alpha a(x)|\Big(|\nabla\phi(x)|^2 + Im(\phi(x))\Big)^{\frac{|\alpha|}{2}-k}. \qquad (3)$$

L. Hörmander, The analysis of PD operators, Springer, 1980.

Lemma (quadratic case, asymptotic expansion)

Let $a \in C_c^\infty(\mathbb{R}^d)$ *and* A *- real, non-singular, symmetric matrix. Then*

$$\Big| \int_{\mathbb{R}^d} a(x) \exp\left(\frac{i}{\epsilon}\frac{1}{2}(Ax,x)\right) dx - \frac{(2\pi i\epsilon)^{d/2}}{(\det(A))^{1/2}} T_k(\epsilon) \Big| \leq C_k(\epsilon \|A^{-1}\|)^{d/2+k} \sum_{|\alpha|\leq 2k+s} \|D^\alpha a\|_{L^2(\mathbb{R}^d)}, \tag{4}$$

with $T_k(\epsilon) = \sum_{j=0}^{k-1} \left(\frac{\epsilon}{2i}\right)^j (A^{-1}D, D)^j \frac{a(0)}{j!}$.

Sketch of proof. Define $\alpha_\epsilon(x) := \exp\left(\frac{i}{\epsilon}\frac{1}{2}(Ax,x)\right)$.

By Parseval identity,

$$\int_{\mathbb{R}^d} a(x)\alpha_\epsilon(x)\, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{a}(\xi)\widehat{\alpha}_\epsilon(-\xi)\, d\xi.$$

$$\widehat{\alpha}_\epsilon(\xi) = \frac{(2\pi i\epsilon)^{d/2}}{|\det(A)|^{1/2}} \exp\left(\frac{\pi i}{4}\mathrm{sgn}(A)\right) \exp\left(-\frac{i\epsilon}{2}(A^{-1}\xi,\xi)\right).$$

Taylor expansion:

$$\boxed{\exp\Big(-\frac{i\epsilon}{2}(A^{-1}\xi,\xi)\Big) = \sum_{j=0}^{k-1}\frac{1}{j!}\Big(-\frac{i\epsilon}{2}(A^{-1}\xi,\xi)\Big)^j + \frac{1}{k!}\Big(-\frac{i\epsilon}{2}(A^{-1}\xi,\xi)\Big)^k \exp(i\theta)}.$$

Use the identity:

$$\sum_{j=0}^{k-1}\frac{1}{j!}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{a}(\xi)\Big(\frac{i\epsilon}{2}(A^{-1}i\xi,i\xi)\Big)^j\,d\xi = \sum_{j=0}^{k-1}\frac{1}{j!}\Big(\frac{i\epsilon}{2}\Big)^j((A^{-1}\nabla,\nabla)^j a)(0) = T_k(\epsilon).$$

The error term:

$$Error \leq \frac{(2\pi\epsilon)^{d/2}}{|\det(A)|^{1/2}}\frac{1}{k!}\Big(\frac{\epsilon\|A^{-1}\|}{2}\Big)^k\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{a}(\xi)||\xi|^{2k}\,d\xi.$$

$$\int_{\mathbb{R}^d}|\widehat{a}(\xi)||\xi|^{2k}\,d\xi \leq 2\pi^{d/2}\omega_d^{1/2}\|a\|_{L^2(\mathbb{R}^d)} + 2\pi^{d/2}\omega_d^{1/2}\frac{1}{\sqrt{2s-d}}\|D^{2k+s}a\|_{L^2(\mathbb{R}^d)}.$$

Lemma

Assume $\phi : \mathbb{R}^d \to \mathbb{R}$ *is smooth.*

i. *Suppose* $\nabla\phi(0) \neq 0$*. Then there exists a smooth function* $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ *s.t.*

$$\begin{cases} \overrightarrow{\varphi}(0) = 0, D\overrightarrow{\varphi}(0) = I \\ \phi(\overrightarrow{\varphi}(x)) = \phi(0) + \nabla\phi(0)\cdot x, \quad |x| \text{ small.} \end{cases}$$

ii. *(Morse Lemma) Suppose that* $\nabla\phi(0) = 0$, $\det(D^2\phi(0)) \neq 0$*. Then there exists a smooth function* $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ *s.t.*

$$\begin{cases} \overrightarrow{\varphi}(0) = 0, D\overrightarrow{\varphi}(0) = I \\ \phi(\overrightarrow{\varphi}(x)) = \phi(0) + \frac{1}{2}x\cdot D^2\phi(0)x, \quad |x| \text{ small.} \end{cases}$$

Sketch of proof. i. Set $v_d = D\phi(0) \neq 0$ and add $v_1, \cdots, v_{d-1}$ s.t. $(v_i)_{i=1,\cdots,d}$ orthogonal basis in $\mathbb{R}^d$.

Apply Implicit Functions Theorem to

$$f(x,y) = (v_1\cdot(x-y), \cdots, v_{d-1}\cdot(x-y), \phi(y) - \phi(0) - v_d\cdot x).$$

Since $D_y f(0,0) \neq 0$, there exists $\overrightarrow{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $\overrightarrow{\varphi}(0) = 0$ and $f(x, \overrightarrow{\varphi}(x)) = 0$ for small $|x|$.

In particular, $\phi(\overrightarrow{\varphi}(x)) = \phi(0) + v_d\cdot x$, $(\overrightarrow{\varphi}(x) - x)\cdot v_i = 0, i = 1, \cdots, d-1.$

By differentiation in x, $(D\overrightarrow{\varphi}(0) - I) \cdot v_i = 0$, $i = 1, \cdots, d$. Then $D\overrightarrow{\varphi}(0) = I$.

ii. Since $D\phi(0) = 0$, $\phi(x) = \phi(0) + \frac{1}{2}(A(x)x, x)$, with $A(x) = 2\int_0^1 (1-t)D^2\phi(tx)\,dt$, with $A(0) = D^2\phi(0)$.

$A(0)$-non-singular. For small $|x|$, $A(x)$-non-singular.

By a rotation, assume $A(0)$-diagonal.

By induction, for each $m = 0, \cdots, d$, construct $\overrightarrow{\varphi}_m : \mathbb{R}^d \to \mathbb{R}^d$ s.t.

$$\begin{cases} \overrightarrow{\varphi}_m(0) = 0, \quad D\overrightarrow{\varphi}_m(0) = I \\ \phi(\overrightarrow{\varphi}_m(x)) = \phi(0) + \frac{1}{2}\sum_{i=1}^{m} \phi_{x_i, x_i}(0)x_i^2 + \frac{1}{2}\sum_{i,j=m+1}^{d} a_{ij}^m(x)x_i x_j. \end{cases} \tag{5}$$

This means $a_{ij}^m(0) = \phi_{x_i, x_j}(0)$. In particular $a_{m+1,m+1}^m(x) \neq 0$ for small $|x|$.

For $m = 0$, $A_0(x) = A(x)$ and $\overline{\varphi}_0(x) = I$.

Assume $\overrightarrow{\varphi}_m$ and A^m defined. Define the mapping $\Pi^{m+1} : \mathbb{R}^d \to \mathbb{R}^d$, $\Pi^{m+1}(y) = x$ by $\Pi^{m+1}(y) =$

$$\left(y_1, \cdots, y_m, \left(\frac{a_{m+1,m+1}^m(y)}{\phi_{x_{m+1}, x_{m+1}}(0)}\right)^{1/2}\left(y_{m+1} + \sum_{j=m+2}^{d} \frac{a_{m+1,j}^m(y)}{a_{m+1,m+1}^m(y)} y_j\right), y_{m+2}, \cdots, y_d\right).$$

Then

$$\phi(\overrightarrow{\varphi}_m(y)) = \phi(0) + \frac{1}{2}\sum_{j=1}^{m+1} \phi_{x_j,x_j}(0)x_j^2 + \frac{1}{2}\sum_{i,j=m+2}^{d} b_{ij}^{m+1}(y)x_i x_j,$$

$$b_{ij}^{m+1}(y) = \begin{cases} a_{ij}^m(y) - \frac{a_{m+1,i}^m(y)a_{m+1,j}^m(y)}{a_{m+1,m+1}^m(y)}, & i,j = m+2,\cdots,d \\ 0, \text{ otherwise.} \end{cases} \qquad (6)$$

Since $D^2\phi(0)$ -diagonal, $\Pi^{m+1}(0) = 0$, $D\Pi^{m+1}(0) = I$.

Then, $(\Pi^{m+1})^{-1}$ well defined.

Set $A^{m+1} = B^{m+1} \circ (\Pi^{m+1})^{-1}$,
$\overrightarrow{\varphi}_{m+1} = \overrightarrow{\varphi}_m \circ (\Pi^{m+1})^{-1}$.

Partition of unity argument:
Suppose $\nabla\phi$ vanishes within the support of a only at a finite number of points, $x_1, \cdots, x_N$ and that $D^2\phi(x_j)$ is non-singular, $\forall j = 1, \cdots, N$.
Consider $\varsigma \in C_c^\infty(\mathbb{R}^d)$ which vanishes near each x_i. Then

$$\Big|\int_{\mathbb{R}^d} a(x)\varsigma(x)\exp\left(\frac{i\phi(x)}{\epsilon}\right)dx\Big| = O(\epsilon^m), \quad \forall m \in \mathbb{N}.$$

$$\Big|\int_{\mathbb{R}^d} a(x)(1-\varsigma(x))\exp\left(\frac{i\phi(x)}{\epsilon}\right)dx - \sum_{j=1}^{N}\exp(i\phi(x_k)/\epsilon)\frac{(2\pi\epsilon)^{d/2}}{|\det(D^2\phi(x_j))|^{1/2}}\exp\left(\frac{i\pi}{4}\text{sign}(D^2\phi(x_j))\right)a(x_j)\Big| = O(\epsilon^{d/2+1}). \tag{7}$$

Lemma (higher order singularities, van der Corput)

Suppose $\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0$ and $\phi^{(k)}(x_0) \neq 0$. Then, if a is supported in a sufficiently small neighborhood of x_0, $I_\epsilon \sim \epsilon^{1/k}\sum_{j=0}^{\infty} a_j\epsilon^{j/k}$.

- The behavior of the Bessel functions at infinity

$$J_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \exp(ir\sin(\theta))\exp(-im\theta)\,d\theta.$$

$\epsilon = 1/r$, $\phi(\theta) = \sin(\theta)$. Singular points: $\theta_1 = \pi/2$, $\theta_2 = 3\pi/2$ and $\phi''(\theta_i) = \pm 1$.

$$J_m(r) = O(r^{-1/2}), \text{ as } r \to \infty.$$

- Riemann singularity

$$\int_0^1 \exp(i\xi x)\exp(i/x)x^{-\gamma}\,dx = \sqrt{\pi i}\exp(2i\xi^{1/2})\xi^{-3/4+\gamma/2} + O(\xi^{-1+\gamma/2}),$$

as $\xi \to \infty$, $0 \le \gamma < 2$.
$\phi(x) = x^{-1} + x\xi$, whose singular point is $x_0 = \xi^{-1/2}$, $\phi''(x_0) = 2\xi^{3/2}$

- Airy function

$$Ai(-x) = \frac{1}{\pi}\int_0^{\infty} \exp\left(i\left(\frac{1}{3}\xi^3 - x\xi\right)\right)d\xi = \pi^{-1/2}x^{-1/4}\exp\left(i\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)\right) + o(x^{-1/4}).$$

$\phi(\xi) = \xi^3/3 - \xi x$, whose critical point is $\xi_0 = x^{1/2}$.

Stationary phase method applied to the wave equation (1)

$$\phi(x,t,y,\eta) = p(y) + \eta\cdot(x-y) + t|\eta|.$$

The set of stationary points $S = \{(y,\eta) : x = y - t\frac{\nabla p(y)}{|\nabla p(y)|}, \eta = \nabla p(y)\}$.

The Hessian matrix

$$D^2_{y,\eta}\phi = \begin{pmatrix} D^2 p(y) & -I \\ -I & \frac{t}{|\eta|}(I - \frac{\eta\otimes\eta}{|\eta|^2}) \end{pmatrix}.$$

$$\det(D^2_{y,\eta}\phi) = (-1)^d \det\left(I - \frac{t}{|\nabla p}|D^2 p(I - \frac{\nabla p\otimes\nabla p}{|\nabla p|^2})\right) = (-1)^d \prod_{i=1}^{d-1}(1 - tk_i(y)),$$

$k_i(y)$- the principal curvatures.

For t sufficiently small, use Implicit Functions Theorem to find $y_0 = y_0(x,t)$, $\eta_0 = \eta_0(x,t)$ the singular points.