# On a generalisation of compactness by compensation in the $L^p$–$L^q$ setting


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October 11, 2013

## Motivation - Maxwell's equations

Let $\Omega \subseteq \mathbf{R}^3$. Denote by E and H the electric and magnetic field, and by D and B the electric and magnetic induction. Let $\rho$ denote the charge, and j the current density. Maxwell's system of equations reads:

$$\begin{aligned} \partial_t \mathsf{B} + \operatorname{rot} \mathsf{E} &= \mathsf{G}, \\ \operatorname{div} \mathsf{B} &= 0, \\ \partial_t \mathsf{D} + \mathsf{j} - \operatorname{rot} \mathsf{H} &= \mathsf{F}, \\ \operatorname{div} \mathsf{D} &= \rho. \end{aligned}$$

Assume that properties of the material can be expressed by following linear constitutive equations:

$$\mathsf{D} = \boldsymbol{\epsilon}\mathsf{E}, \ \mathsf{B} = \boldsymbol{\mu}\mathsf{H}.$$

The energy of electromagnetic field at time $t$ is given by:

$$T(t) = \frac{1}{2}\int_\Omega (\mathsf{D}\cdot\mathsf{E} + \mathsf{B}\cdot\mathsf{H})d\mathbf{x}.$$

It's natural to consider

$$\mathsf{D}, \mathsf{B} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2_{\mathsf{div}}(\Omega; \mathbf{R}^3)),$$

$$\mathsf{E}, \mathsf{H} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2_{\mathsf{rot}}(\Omega; \mathbf{R}^3)),$$

$$\mathsf{J} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^2(\Omega; \mathbf{R}^3)), \quad \mathsf{F}, \mathsf{G} \in \mathrm{L}^2([0,T]; \mathrm{L}^2(\Omega; \mathbf{R}^3)).$$

Let us consider a family of problems:

$$\partial_t \mathsf{B}^n + \operatorname{rot} \mathsf{E}^n = \mathsf{G}^n,$$

$$\partial_t \mathsf{D}^n + \mathsf{J}^n - \operatorname{rot} \mathsf{H}^n = \mathsf{F}^n,$$

with constitution equations:

$$\mathsf{D}^n = \boldsymbol{\epsilon}^n \mathsf{E}^n, \quad \mathsf{B}^n = \boldsymbol{\mu}^n \mathsf{H}^n, \quad \mathsf{J}^n = \boldsymbol{\sigma}^n \mathsf{E}^n.$$

What can we say about energy $T(t)$ if we know

$$T^n(t) = \frac{1}{2} \int_\Omega (\mathsf{D}^n \cdot \mathsf{E}^n + \mathsf{B}^n \cdot \mathsf{H}^n) d\mathbf{x}.$$

# Div-rot lemma in $\mathrm{L}^2$

**Theorem 1.** *Assume that* $\Omega$ *is open and bounded subset of* $\mathbf{R}^3$*, and that it holds:*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } \mathrm{L}^2(\Omega; \mathbf{R}^3),$$

$$\operatorname{rot} \mathbf{u}_n \text{ bounded in } \mathrm{L}^2(\Omega; \mathbf{R}^3), \ \operatorname{div} \mathbf{v}_n \text{ bounded in } \mathrm{L}^2(\Omega).$$

*Then*

$$\mathbf{u}_n \cdot \mathbf{v}_n \rightharpoonup \mathbf{u} \cdot \mathbf{v}$$

*in the sense of distributions.* ∎

# Quadratic theorem

**Theorem 2.** *(**Quadratic theorem**) Assume that $\Omega \subseteq \mathbf{R}^d$ is open and that $\Lambda \subseteq \mathbf{R}^r$ is defined by*

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbf{R}^r \,:\, (\exists\, \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) \quad \sum_{k=1}^{d} \xi_k \mathbf{A}^k \boldsymbol{\lambda} = 0 \right\},$$

*where $Q$ is a real quadratic form on $\mathbf{R}^r$, which is nonnegative on $\Lambda$, i.e.*

$$(\forall\, \boldsymbol{\lambda} \in \Lambda) \quad Q(\boldsymbol{\lambda}) \geqslant 0 \,.$$

*Furthermore, assume that the sequence of functions $(\mathbf{u}_n)$ satisfies*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \textit{weakly in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{R}^r)\,,$$

$$\left( \sum_k \mathbf{A}^k \partial_k \mathbf{u}_n \right) \quad \textit{relatively compact in} \quad \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{R}^q)\,.$$

*Then every subsequence of $(Q \circ \mathbf{u}_n)$ which converges in distributions to it's limit $L$, satisfies*

$$L \geqslant Q \circ \mathbf{u}$$

*in the sense of distributions.* ∎

The most general version of the classical $L^2$ results has recently been proved by E. Yu. Panov (2011):

Assume that the sequence $(\mathbf{u}_n)$ is bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$, $2 \le p < \infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k(\mathbf{A}^k \mathbf{u}_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl}(\mathbf{B}^{kl}\mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $W^{-1,-2;q}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$, where $m \times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficents belonging to $L^{2\bar{q}}(\mathbf{R}^d)$, $\bar{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $C(\mathbf{R}^d)$ if $p = 2$.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x}) = \Big\{ \boldsymbol{\lambda} \in \mathbf{C}^r \,\Big|\, (\exists \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{0\}) : \tag{1}$$

$$\Big( i \sum_{k=1}^{\nu} \xi_k \mathbf{A}^k(\mathbf{x}) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l \mathbf{B}^{kl}(\mathbf{x}) \Big) \boldsymbol{\lambda} = \mathbf{0}_m \Big\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x}) \boldsymbol{\lambda} \cdot \boldsymbol{\eta}, \tag{2}$$

where $\mathbf{Q} \in L^{\bar{q}}_{loc}(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p > 2$ and $\mathbf{Q} \in C(\mathbf{R}^d; \mathrm{Sym}_r)$ if $p = 2$.
Finally, let $q(\mathbf{x}, \mathbf{u}_n, \mathbf{u}_n) \rightharpoonup \omega$ weakly in the space of distributions.

# Result by Panov

The following theorem holds

**Theorem 3.** *[P, 2011] Assume that* $(\forall \boldsymbol{\lambda} \in \Lambda(\mathbf{x}))\ q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq 0$ *(a.e.* $\mathbf{x} \in \mathbf{R}^d$*) and* $\mathbf{u}_n \rightharpoonup \mathbf{u}$*, then* $q(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{x})) \leq \omega$*.* ∎

The connection between $q$ and $\Lambda$ given in the previous theorem, we shall call *the consistency condition*.

# H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the $L^p - L^q$ context.

M. Lazar and D. Mitrović (2012) extended and applied them on a velocity averaging problem.

We need Fourier multiplier operators with symbols defined on a manifold $\mathrm{P}$ determined by $d$-tuple $\boldsymbol{\alpha} \in (\mathbf{R}^+)^d$:

$$\mathrm{P} = \Big\{\boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{l\alpha_k} = 1\Big\},$$

where $l$ is the smallest number such that $l\alpha_k > d$ for each $k$. In order to associate an $L^p$ Fourier multiplier to a function defined on $\mathrm{P}$, we extend it to $\mathbf{R}^d \backslash \{0\}$ by means of the projection

$$(\pi_{\mathrm{P}}(\boldsymbol{\xi}))_j = \xi_j \left(|\xi_1|^{l\alpha_1} + \cdots + |\xi_d|^{l\alpha_d}\right)^{-1/l\alpha_j}, \quad j = 1, \ldots, d.$$

**Theorem 4.** *[LM, 2012] Let* $(u_n)$ *be a bounded sequence in* $L^s(\mathbf{R}^d)$, $s > 1$, *and let* $(v_n)$ *be a bounded sequence of uniformly compactly supported functions in* $L^\infty(\mathbf{R}^d)$. *Then, after passing to a subsequence (not relabelled), for any* $\bar{s} \in (1, s)$ *there exists a continuous bilinear functional* $B$ *on* $L^{\bar{s}'}(\mathbf{R}^d) \otimes C^d(\mathrm{P})$ *such that for every* $\varphi \in L^{\bar{s}'}(\mathbf{R}^d)$ *and* $\psi \in C^d(\mathrm{P})$ *it holds*

$$B(\varphi, \psi) = \lim_{n\to\infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \big(\mathcal{A}_{\psi_\mathrm{P}} v_n\big)(\mathbf{x}) d\mathbf{x}\,,$$

*where* $\mathcal{A}_{\psi_\mathrm{P}}$ *is the Fourier multiplier operator on* $\mathbf{R}^d$ *associated to* $\psi \circ \pi_\mathrm{P}$. ∎

**Corollary 1.** *[LM, 2012] The bilinear functional* $B$ *defined in the previous theorem can be extended by continuity to a continuous functional on* $L^{\bar{s}'}(\mathbf{R}^d; C^d(\mathrm{P}))$. ∎

We need the following extension of the results given above.

**Theorem 5.** *Let* $(u_n)$ *be a bounded sequence in* $L^p(\mathbf{R}^d)$, $p > 1$, *and let* $(v_n)$ *be a bounded sequence of uniformly compactly supported functions in* $L^q(\mathbf{R}^d)$, $1/q + 1/p < 1$. *Then, after passing to a subsequence (not relabelled), for any* $\bar{s} \in (1, \frac{pq}{p+q})$ *there exists a continuous bilinear functional* $B$ *on* $L^{\bar{s}'}(\mathbf{R}^d) \otimes C^d(\mathrm{P})$ *such that for every* $\varphi \in L^{\bar{s}'}(\mathbf{R}^d)$ *and* $\psi \in C^d(\mathrm{P})$, *it holds*

$$B(\varphi, \psi) = \lim_{n\to\infty} \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) \big(\mathcal{A}_{\psi_{\mathrm{P}}} v_n\big)(\mathbf{x}) d\mathbf{x}\,,$$

*where* $\mathcal{A}_{\psi_{\mathrm{P}}}$ *is the Fourier multiplier operator on* $\mathbf{R}^d$ *associated to* $\psi \circ \pi_{\mathrm{P}}$. *The bilinear functional* $B$ *can be continuously extended as a linear functional on* $L^{\bar{s}'}(\mathbf{R}^d; C^d(\mathrm{P}))$.

∎

# Localisation principle

**Lemma**

Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$ and $L^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{u}$ and $\mathbf{0}$ in the sense of distributions.

Furthermore, assume that sequence $(\mathbf{u}_n)$ satisfies:

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial_k^{\alpha_k}(\mathbf{A}^k \mathbf{u}_n) \ \to \mathbf{0} \text{ in } W^{-1,p}(\Omega;\mathbf{R}^m), \tag{3}$$

where either $\alpha_k \in \mathbf{N}$, $k = 1,\dots,d$ or $\alpha_k > d$, $k = 1,\dots,d$, and elements of matrices $\mathbf{A}^k$ belong to $L^{\bar{s}'}(\mathbf{R}^d)$, $\bar{s} \in (1, \frac{pq}{p+q})$.

Finally, by $\boldsymbol{\mu}$ denote a matrix $H$-distribution corresponding to subsequences of $(\mathbf{u}_n)$ and $(\mathbf{v}_n - \mathbf{v})$. Then the following relation holds

$$\Big(\sum_{k=1}^{d}(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}.$$

# Proof of the lemma

Assume, without loosing any generality, that $\mathbf{v} = \mathbf{0}$. Denote by $\mathcal{B}_\psi$ the Fourier multiplier operator with the symbol

$$(\psi \circ \pi_{\mathrm{P}})(\boldsymbol{\xi}) \frac{(1 - \theta(\boldsymbol{\xi}))}{(|\xi_1|^{l\alpha_1} + \cdots + |\xi_d|^{l\alpha_d})^{1/l}}.$$

According to [LM 2012, Lemma 5], for any $\psi \in C^d(\mathrm{P})$ and any $\hat{s} > 1$, the multiplier operator $\mathcal{B}_\psi : L^2(\mathbf{R}^d) \cap L^{\hat{s}}(\mathbf{R}^d) \to W^{\boldsymbol{\alpha};\hat{s}}(\mathbf{R}^d)$, is bounded (with $L^{\hat{s}}$ norm considered on the domain of operator $\mathcal{B}_\psi$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$)
Take a test function $\mathbf{g}_n$ given by:

$$\mathbf{g}_n(\mathbf{x}) = \mathcal{B}_\psi\big(\phi \mathbf{v}_n\big)(\mathbf{x}),$$

where $\psi \in \mathrm{C}^d(\mathrm{P})$ and $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ and multiply with $\mathbf{G}_n$.

# Proof of the lemma - continued

We get

$$\int_{\mathbf{R}^d} \mathbf{G}_n \cdot \mathbf{g}_n d\mathbf{x} = \int_{\mathbf{R}^d} \sum_{k=1}^{d} \mathbf{A}^k \mathbf{u}_n \cdot \mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{(1-\theta(\boldsymbol{\xi}))(2\pi i\xi_k)^{\alpha_k}}{\left(|\xi_1|^{l\alpha_1}+\cdots+|\xi_d|^{l\alpha_d}\right)^{1/l}}}(\phi\mathbf{v}_n)d\mathbf{x}$$

$$= \int_{\mathbf{R}^d} \sum_{k=1}^{d} \mathbf{A}^k \mathbf{u}_n \cdot \mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{(2\pi i\xi_k)^{\alpha_k}}{\left(|\xi_1|^{l\alpha_1}+\cdots+|\xi_d|^{l\alpha_d}\right)^{1/l}}}(\phi\mathbf{v}_n)d\mathbf{x} +$$

$$+ \int_{\mathbf{R}^d} \sum_{k=1}^{d} \mathbf{A}^k \mathbf{u}_n \cdot \mathcal{A}_{(\psi\circ\pi_{\mathrm{P}})(\boldsymbol{\xi})\frac{\theta(\boldsymbol{\xi})(2\pi i\xi_k)^{\alpha_k}}{\left(|\xi_1|^{l\alpha_1}+\cdots+|\xi_d|^{l\alpha_d}\right)^{1/l}}}(\phi\mathbf{v}_n)d\mathbf{x}.$$

Taking into account the preceeding theorem and strong precompactess of sequence $(\mathbf{G}_n)$, we get

$$\Big(\sum_{k=1}^{d}(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol{\mu} = \mathbf{0}.$$

# Strong consistency condition

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{ \boldsymbol{\mu} \in \ L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r : \Big( \sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}_m \Big\},$$

where the given equality is understood in the sense of $L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^m$.

**Definition**

We say that set $\Lambda_{\mathcal{D}}$, bilinear form $q$ from (2) and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r], \boldsymbol{\mu}_j \in L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \dots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq \mathbf{0}, \quad \phi \in L^{\bar{s}}(\mathbf{R}^d; \mathbf{R}^+_0).$$

Let us assume that coefficients of the bilinear form $q$ from (2) belong to space $L^t_{loc}(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

**Theorem 6.** *Assume that sequences* $(\mathbf{u}_n)$ *and* $(\mathbf{v}_n)$ *are bounded in* $L^p(\mathbf{R}^d; \mathbf{R}^r)$ *and* $L^q(\mathbf{R}^d; \mathbf{R}^r)$*, respectively, and converge toward* $\mathbf{u}$ *and* $\mathbf{v}$ *in the sense of distributions.*
*Assume that* (3) *holds and that*

$$q(\mathbf{x}; \mathbf{u}_n, \mathbf{v}_n) \rightharpoonup \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

*If the set* $\Lambda_{\mathcal{D}}$*, the bilinear form* (2)*, and matrix* $H$*-distribution* $\boldsymbol{\mu}$*, corresponding to subsequences of* $(\mathbf{u}_n - \mathbf{u})$ *and* $(\mathbf{v}_n - \mathbf{v})$*, satisfy the strong consistency condition, then*

$$q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \leq \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

∎

# Proof of the theorem

Let us abuse the notation by denoting $\mathbf{u}_n = \mathbf{u}_n - \mathbf{u} \rightharpoonup \mathbf{0}$ and $\mathbf{v}_n = \mathbf{v}_n - \mathbf{v} \rightharpoonup \mathbf{0}$.
According to the theorem on existence of H-distributions, for any non-negative $\phi \in \mathcal{D}(\mathbf{R}^d)$ it holds

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} \mathbf{Q}\mathbf{u}_n \cdot \mathbf{v}_n \phi \, d\mathbf{x} = \langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle,$$

where $\boldsymbol{\mu}$ is matrix H-distribution corresponding to (subsequences of) $\mathbf{u}_n, \mathbf{v}_n \rightharpoonup \mathbf{0}$. Since, according to the localisation principle, for every fixed $k \in \{1, \dots, r\}$, the $r$-tuple $\boldsymbol{\mu}_k$ belongs to $\Lambda_{\mathcal{D}}$, we conclude from the strong consistency condition that

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq 0.$$

From the fact that

$$q(\mathbf{x}; \mathbf{u}_n, \mathbf{v}_n) \rightharpoonup \omega - q(\mathbf{x}; \mathbf{u}, \mathbf{v}) \geq 0 \ \text{ in } \ \mathcal{D}'(\mathbf{R}^d),$$

the statement of the theorem follows.

# Application to the parabolic type equation

Now, let us consider the non-linear parabolic type equation

$$L(u) = \partial_t u - \mathsf{div}\,\mathsf{div}\,(g(t, \mathbf{x}, u)\mathbf{A}(t, \mathbf{x})),$$

on $(0, \infty) \times \Omega$, where $\Omega$ is an open subset of $\mathbf{R}^d$. We assume that

$$u \in L^p((0,\infty) \times \Omega), \quad g(t, \mathbf{x}, u(t, \mathbf{x})) \in L^q((0,\infty) \times \Omega), \quad 1 < p, q,$$
$$\mathbf{A} \in L^s_{loc}((0,\infty) \times \Omega)^{d \times d}, \quad \text{where} \;\; 1/p + 1/q + 1/s < 1,$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \in \mathbf{R}^d \setminus \{\mathbf{0}\}, \quad (a.e.(t, \mathbf{x}) \in (0,\infty) \times \Omega).$$

Furthermore, assume that $g$ is a Carathèodory function and non-decreasing with respect to the third variable.

Then we have the following theorem.

**Theorem 7.** *Assume that sequences* $(u_r)$ *and* $g(\cdot, u_r)$ *are such that* $u_r, g(u_r) \in L^2(\mathbf{R}^+ \times \mathbf{R}^d)$ *for every* $r \in \mathbf{N}$*; assume that they are bounded in* $L^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p \in (1, 2]$*, and* $L^q(\mathbf{R}^+ \times \mathbf{R}^d)$, $q > 2$*, respectively, where* $1/p + 1/q < 1$*; furthermore, assume* $u_r \rightharpoonup u$ *and, for some,* $f \in W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d)$*, the sequence*

$$L(u_r) = f_r \to f \quad \textit{strongly in } W^{-1,-2;p}(\mathbf{R}^+ \times \mathbf{R}^d).$$

*Under the assumptions given above, it holds*

$$L(u) = f \ \text{ in } \ \mathcal{D}'(\mathbf{R}^+ \times \mathbf{R}^d).$$

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# References


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