A Matlab Interface for Control of Waves

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Constant coefficient $1-D$ wave equation

Consider

$$(*)\begin{cases} u_{tt}-u_{xx}=0 & 0<x<1, \quad 0<t<T \\ u(0,t)=u(1,t)=0 & 0<t<T \\ u(x,0)=u^0(x), u_t(x,0)=u^1(x) & 0<x<1 \end{cases}$$

Where $u=u(x,t)$ describes the displacement of a vibrating string occupying the interval (0,1).

Boundary controllability

The system is controllable if and only if, for any $(y^0(x), y^1(x)) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $v(t) \in L^2(0,T)$ such that the solution of the controlled wave equation

$$(**) \begin{cases} y_{tt} - y_{xx} = 0 & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0, \quad y(1,t) = v(t) & 0 < t < T \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & 0 < x < 1 \end{cases}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1 \tag{1}$$

Boundary controllability

It is well known that, when $T \geq 2$ and given $(y^0(x), y^1(x)) \in L^2(0,1) \times H^{-1}(0,1)$ the control $v(t) \in L^2(0,T)$ of minimal norm for which (1) holds is of the form

$$v(t) = \hat{u}_x(1,t) \tag{2}$$

where $\hat{u}$ is the solution of (*) corresponding to initial data $(\hat{u}^0, \hat{u}^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(u^0, u^1) = \frac{1}{2} \int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - \langle y^1, u^0 \rangle_{H^{-1} \times H_0^1} \tag{3}$$

Structure of the platform

The platform allows you to choose among the four different problems that can be solved:

- Control of the $1-D$ wave equation.
- Control of the $2-D$ wave equation.
- Stabilization of the $1-D$ wave equation.
- Stabilization of the $2-D$ wave equation.

$\rightarrow$ The first two options compute approximations of the boundary control of the wave equation by solving the discrete version of the linear systems (*) and (**) with a **conjugate gradient algorithm (HUM)**.

$\rightarrow$ Three different spatial semi-discretization schemes are available for solving the $1-D$ wave equation in the control problems: **finite differences, finite elements and mixed finite elements.**

Objective

Discretize the continuous model, then compute the control of the discrete system and use it as a numerical approximation of the continuous one.

Difficulties

- The numerical approximation of the control is not an easy problem.
- In general, any discrete dynamics associated to the wave equation generates spurious high-frequency oscillations that do not exist at the continuous level.
- Moreover, a numerical dispersion phenomenon appears and the velocity of propagation of some high frequency numerical waves may possibly converge to zero when the mesh size h does.
- In this case, the controllability property for the discrete system will not be uniform, as $h \to 0$, for a fixed time T and, consequently, there will be initial data for which the corresponding controls of the discrete model will diverge as $h \to 0$.

Solutions implemented in the platform

1. Filter the high frequencies:
   - **Fourier filtering.** It consists in removing from the solution of the wave equation the part which corresponds to the numerical high frequencies. Filter parameter: $0 < \gamma < 4$.
   - **Bi-Grid algorithm.** It introduces two grids. A fine one with step h and a coarse one with step $2h$. The main idea is to consider a space discretization scheme of the wave equations in the fine grid but taking the initial (or final) data in the coarse one. In this way, the high frequencies associated to the fine grid are eliminated.
2. Other numerical schemes:
   - **Mixed finite elements.** In this case the velocity of propagation of the high frequency numerical waves becomes large, as $h \to 0$. Thus, the minimal velocity of propagation of waves in this semi-discrete system is the same as in the continuous one.

Space discretization for the wave equation

We introduce a uniform mesh of the space interval (a, b) with step $h = (b - a)/(n + 1)$, where n is the number of internal points or nodes.
The wave equation is then replaced by semi-discrete systems of the form:

$$MU'' + KU = F$$

where M is the mass matrix, K is the stiffness matrix, F is the vector of applied forces and $U = (U_1, \dots, U_n)^T$ is the displacement vector at the nodes.

Space discretization for the wave equation

The following choices of K and M are considered:

$$
\begin{aligned}
K &= -\Delta_h := \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \\
M &= I + rh^2\Delta_h
\end{aligned}
$$

Space discretization for the wave equation

The parameter r allows to choose three particularly interesting models:

1. $r = 0$. In this case $M = I$ and we obtain the finite difference method.
2. $r = 1/6$. In this case M is known as *consistent mass matrix* and the system corresponds to the classical finite element method.
3. $r = 1/4$. In this case M is the matrix associated to the mixed finite element method.

Time discretization

The above semi-discrete system can be written in a general form as follows

$$MU'' + CU' + KU = F$$

where M is the mass matrix, C is the viscous damping matrix, K is the stiffness matrix, F is the vector of applied forces and U, U', U'' are the displacement, velocity and acceleration vectors, respectively.

Time discretization

For solve this, we use *Newmark method*. Thus, we substitute the previous equation by the following

$$MA_{j+1} + CV_{j+1} + KU_{j+1} = F_{j+1}$$
$$U_{j+1} = U_j + \Delta t V_j + \frac{\Delta t^2}{2}[(1-2\beta)A_j + 2\beta A_{j+1}]$$
$$V_{j+1} = V_j + \Delta t[(1-\gamma)A_j + \gamma A_{j+1}]$$
$$U_0 = U(0), \quad V_0 = U'(0), \quad MA_0 = F_0 - KU_0$$

The parameters β and γ determine the stability and accuracy characteristics of the algorithm under consideration. Two particular cases

1. $\beta = 0$ and $\gamma = 1/2$. Central difference.
2. $\beta = 1/4$ and $\gamma = 1/2$. Trapezoidal rule.

Time discretization

The choice for the time discretization scheme depends on the chosen scheme for the discretization in space:

1. Central difference in space.
   - $\Delta t = h$ will produce the exact solution of the wave equation at the nodes.
   - We require $\Delta t / h \leq 1$.
2. Mixed finite element in space. In this case the velocity of propagation of the high frequency numerical waves becomes large as $h \to 0$. We require $\Delta t \leq Ch^2$ for some $C > 0$.

Examples using the platform