Stability Analysis for Transmission Problem in Thermoelasticity

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Overview

$$\boxed{\text{heat}} \leftrightarrow \boxed{\text{elasticity}}$$

Dynamical behavior: 1-d linear form

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta \theta_x(x,t) = 0, \ t > 0, x \in (0,\ell), \\ \theta_t(x,t) + \gamma q_x(x,t) + \delta u_{tx}(x,t) = 0, \ t > 0, x \in (0,\ell), \\ \tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0, \ t > 0, x \in (0,\ell), \end{cases} \tag{1}$$

where $u(x,t)$ is the displacement at time t, position x; $\theta(x,t)$ is the temperature difference to a fixed reference temperature and $q(x,t)$ is the heat flux. Assume that the parameters α, β, γ, δ, k are all positive parameters and $\tau \geq 0$.

Thermoelasticity

the pure propagation of heat

$$\theta_t(x,t) + \gamma q_x(x,t) = 0$$

$$\tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0$$

- $\tau = 0$: Fourier law (classical thermoelasticity)
- $\tau > 0$: Cattaneo Law (thermoelasticity of Lord-Shulman type)

The pure heat conduction are all exponentially stable to zero under these two laws.

When $\tau = 0$, system (1) is a hyperbolic-parabolic coupled system for u and θ.
When $\tau > 0$, system (1) becomes a hyperbolic-hyperbolic coupled one.

Thermoelasticity

Many kinds of thermoelastic systems have been considered, please see for example:

References

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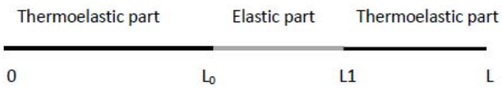
Transmission Problem in Thermoelasticity



- pure elastic body$\rightarrow$ conservative system;
- thermoelastic materials$\rightarrow$ damping mechanism

If the thermal damping mechanism is effective in the whole domain, it always leads to exponential stability for one-dimensional linear systems; While what will happen if it is only effective in a sub-domain? The corresponding mathematical model is called a **transmission problem**.

Previous Results on Its Stability

stability

The stability of various transmission problems on thermoelasticity have been considered. For **one-dimensional system**, the systems are proved to be **exponentially stable, no matter how small the thermoelastic parts are.** See for example: Racke, Rivera et al [1][2][3].

Remark: For the system with nonlinear term or the system with memory, some conditions on nonlinear term or relaxation function must be added to ensure the stability of the whole system.

References

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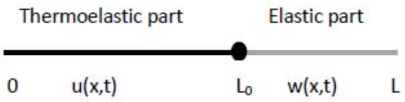
[2]L.H. Fatori, E. Lueders and J.E. Munoz Rivera, Transmission problem for hyperbolic thermoelastic systems, *J. Thermal Stresses*, 2003.
[3] H. D. Fernandez Sare, J. E. Munoz Rivera and R. Racke, Stability for a transmission problem in thermoelasticity with second sound

2. Transmission problem with concentrated mass



$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta\theta_x(x,t) = 0,\ x \in (0, L_0),\ 0 < L_0 < L,\ t > 0, \\ \theta_t(x,t) + \gamma q_x(x,t) + \delta u_{tx}(x,t) = 0,\ x \in (0, L_0),\ t > 0, \\ \tau q_t(x,t) + q(x,t) + k\theta_x(x,t) = 0,\ x \in (0, L_0),\ t > 0, \\ w_{tt}(x,t) - b w_{xx}(x,t) = 0,\ x \in (L_0, L),\ t > 0, \end{cases} \tag{2}$$

where $u(x,t), w(x,t)$ denote the displacements in intervals $(0, L_0)$ and (L_0, L), respectively. $\theta(x,t)$ and $q(x,t)$ for $x \in (0, L_0)$ denote the temperature difference to a fixed reference temperature, and the heat flux.

We mainly consider the case $\tau > 0$, that is, the thermoelasticity of Lord-Shulman type.

Boundary and Transmission Condition

Boundary conditions at $x = 0, L$:

$$\begin{cases} u_x(0,t) = 0, \quad t > 0, \\ \theta(0,t) = 0, \quad t > 0, \\ w(L,t) = 0, \quad t > 0, \end{cases} \tag{3}$$

Transmission conditions at $x = L_0$:

$$\begin{cases} q(L_0,t) = 0, \\ u(L_0,t) = w(L_0,t), \\ -Mu_{tt}(L_0,t) = \alpha u_x(L_0,t) - \beta\theta(L_0,t) - bw_x(L_0,t), \quad M > 0. \end{cases} \tag{4}$$

The initial conditions:

$$\begin{cases} u(.,0) = u_0,\ u_t(.,0) = u_1,\ \theta(.,0) = \theta_0,\ q(.,0) = q_0, \quad in\ (0, L_0), \\ w(.,0) = w_0,\ w_t(.,0) = w_1, \quad in\ (L_0, L). \end{cases} \tag{5}$$

Semigroup Setting

Set

$$V^k[a,b] := \{u \in H^k[a,b] \mid u(a) = 0\},$$
$$\widetilde{V}^k[a,b] := \{u \in H^k[a,b] \mid u(b) = 0\}.$$

State space: $\mathcal{H} := \Big\{(u, v, w, z, \theta, q, p)^T \in$
$H^1[0, L_0] \times L^2(0, L_0) \times \widetilde{V}^1[L_0, L] \times L^2(L_0, L) \times L^2(0, L_0) \times L^2(0, L_0) \times \mathbb{C} \Big| u(L_0) = w(L_0)\Big\}$,
equipped with an inner product, for
$W_j = (u_j, v_j, w_j, z_j, \theta_j, q_j, p_j) \in \mathcal{H},\ j = 1, 2,$

$$\begin{aligned}(W_1, W_2)_{\mathcal{H}} \quad = \quad & \int_0^{L_0} [v_1(x)\overline{v_2(x)} + \alpha u_{1,x}(x)\overline{u_{2,x}(x)} + \frac{\beta}{\delta}\theta_1(x)\overline{\theta_2(x)} \\ & + \frac{\tau\gamma\beta}{k\delta} q_1(x)\overline{q_2(x)}]dx + \int_{L_0}^{L} [z_1(x)\overline{z_2(x)} + bw_{1,x}(x)\overline{w_{2,x}(x)}]dx \\ & + Mp_1\overline{p_2}.\end{aligned}$$

System operator

$$\mathcal{A}\begin{bmatrix} u \\ v \\ w \\ z \\ \theta \\ q \\ p \end{bmatrix} = \begin{pmatrix} v \\ \alpha u_{xx} - \beta\theta_x \\ z \\ bw_{xx} \\ -\gamma q_x - \delta v_x \\ -\frac{1}{\tau}(q + k\theta_x) \\ -\frac{1}{M}[\alpha u_x(L_0) - bw_x(L_0) - \beta\theta(L_0)] \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, v, w, z, \theta, q, p) \in \mathcal{H} \;\middle|\; \begin{array}{l} u \in H^2[0, L_0],\ v \in H^1[0, L_0], \\ w \in \widetilde{V}^2[L_0, L],\ z \in \widetilde{V}^1[L_0, L] \\ \theta \in V^2[0, L_0],\ q \in \widetilde{V}^1[0, L_0] \\ p = v(L_0),\ u_x(0) = 0 \end{array} \right\}.$$

Evolution Equation

System (2)–(5) can be rewritten as an evolutionary equation in $\mathcal{H}$:

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{6}$$

where

$$U(t) = (u, u_t, w, w_t, \theta, q, p)^T$$

$$U(0) = (u^0(x), u^1(x), w^0(x), w^1(x), \theta^0(x), q_0, p_0)^T \in \mathcal{H}.$$

Semigroup Setting

Lemma

Let $\mathcal{A}$ and $\mathcal{H}$ be defined as in above. Then
1) $\mathcal{A}$ is dissipative in $\mathcal{H}$;
2) $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, the spectrum of $\mathcal{A}$ consists of isolated eigenvalues of finite multiplicities only, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.

Theorem

Let $\mathcal{A}$ and $\mathcal{H}$ be defined as above. Then $\mathcal{A}$ generates a C_0 semigroup of contractions $S(t)$ on $\mathcal{H}$.

Spectral Analysis of $\mathcal{A}$

For any $\lambda \in \sigma(\mathcal{A})$, suppose that $(u, v, w, z, \theta, q, p)^T \in \mathcal{D}(\mathcal{A})$ is an eigenvector of $\mathcal{A}$ corresponding to λ.

Eigenvalue Problem: $(\lambda I - \mathcal{A})(u, v, w, z, \theta, q, p)^T = 0$

$$\Updownarrow$$

$$\begin{cases} \lambda^2 u(x) - \alpha u_{xx}(x) + \beta\theta_x(x) = 0, x \in (0, L_0), \\ \lambda\theta(x) + rq_x(x) + \delta\lambda u_x(x) = 0, x \in (0, L_0), \\ (\tau\lambda + 1)q(x) + k\theta_x(x) = 0, x \in (0, L_0), \\ \lambda^2 w(x) - bw_{xx}(x) = 0, x \in (L_0, L), \\ u_x(0) = \theta(0) = w(L) = 0, \\ q(L_0) = 0, u(L_0) = w(L_0), \\ -M\lambda^2 u(L_0) = \alpha u_x(L_0) - \beta\theta(L_0) - bw_x(L_0), \quad M > 0. \end{cases} \tag{7}$$

Spectral analysis of $\mathcal{A}$

Lemma

Let $\mathcal{A}$ and $\mathcal{H}$ be defined as in above. Then there is no eigenvalue on the imaginary axis.

Lemma

If τ satisfies

$$\frac{M}{\tau} = \sqrt{\alpha}\frac{e^{\sqrt{\frac{1}{\alpha}\frac{1}{\tau}}L_0} - e^{-\sqrt{\frac{1}{\alpha}\frac{1}{\tau}}L_0}}{e^{\sqrt{\frac{1}{\alpha}\frac{1}{\tau}}L_0} + e^{-\sqrt{\frac{1}{\alpha}\frac{1}{\tau}}L_0}} - \sqrt{b}\frac{1 + e^{2\sqrt{\frac{1}{b}\frac{1}{\tau}}(L-L_0)}}{1 - e^{2\sqrt{\frac{1}{b}\frac{1}{\tau}}(L-L_0)}}, \tag{8}$$

then $-\frac{1}{\tau}$ is an eigenvalue of $\mathcal{A}$.

Spectral analysis of $\mathcal{A}$

Theorem

Let $\mathcal{A}$ and $\mathcal{H}$ be defined as in above. Then when $\lambda \neq -\frac{1}{\tau}$ with sufficiently large $|\lambda|$, the spectrum of $\mathcal{A}$ has three branches given as follows

$$\lambda_1^n = \frac{1}{\sqrt{\xi_1}}\left(-\widetilde{a}_1 + \frac{(n+\frac{1}{2})\pi i}{L_0}\right) + \mathcal{O}(n^{-2}), \ n \in \mathbb{Z}, \tag{9}$$

$$\lambda_2^n = \frac{1}{\sqrt{\xi_2}}\left(\frac{\widetilde{a}_2}{h} - \frac{(n+\frac{1}{2})\pi i}{L_0}\right) + \mathcal{O}(n^{-2}), \ n \in \mathbb{Z}, \tag{10}$$

$$\lambda_3^n = \frac{n\sqrt{b}\pi i}{L - L_0} + \mathcal{O}(n^{-6}), \ n \in \mathbb{Z}. \tag{11}$$

where $h < 0$ and $\widetilde{a}_j > 0, \ j = 1, 2.$

Stability Analysis

spectral property

- there is no eigenvalue on imaginary axis;
- the imaginary axis is one asymptote of the spectrum;

$$\Downarrow$$

Theorem

The total energy of the system (2)–(5) decays to 0 *asymptotically.*

Polynomial Stability

Lemma

(Borichev and Tomilov [1] and Liu and Rao [2])A C_0 semigroup $e^{t\mathcal{A}}$ of contractions on a Hilbert space satisfies

$$\|e^{t\mathcal{A}}U_0\| \leq Ct^{-\frac{1}{\ell}}\|U_0\|_{\mathcal{D}(\mathcal{A})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{A}),\ t \to \infty$$

for some constant $C > 0$, if the following conditions hold:

1). $\{i\beta | \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$

2). $\lim\limits_{|\beta|\to\infty} \sup \frac{1}{\beta^\ell}\|(i\beta - \mathcal{A})^{-1}\| < \infty.$

Theorem

The $(T(t))_{t\geq 0}$ associated with the system (2)–(5) decays polynomially as

$$\|T(t)U_0\| \leq \frac{C}{t^{\frac{1}{6}}}\|U_0\|_{\mathcal{D}(\mathcal{A})}.$$

Polynomial Stability

Question

Is $\frac{1}{6}$ a sharp estimate for the polynomial decay rate?

By one branch of the spectrum

$$\lambda_3^n = \frac{n\sqrt{b}\pi i}{L - L_0} + \mathcal{O}(n^{-6}), \ n \in \mathbb{Z},$$

Polynomial decay order α must satisfy $\Im\lambda \geq \frac{C}{(\Re\lambda)^\alpha}, \ \lambda \in \sigma(A)$.

Theorem

$\frac{1}{6}$ *is the optimal polynomial decay order.*

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References

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Welcome to Tianjin!

Thank you!