Control of evolution problems involving the fractional Laplace operator

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Fractional laplacian

$$(-\Delta)^s u(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \qquad s \in (0,1)$$

$$c_{n,s} := \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2} \Gamma(1-s)}$$

We analyse the control problem for the fractional wave equation

$$u_{tt} + (-\Delta)^{s+1} u = 0$$

on a bounded $C^{1,1}$ domain Ω of $\mathbb{R}^n$. We focus on the control from a neighborhood of the boundary $\partial\Omega$.

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Fractional Sobolev spaces

Definition

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \,\middle|\, \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}$$

Definition

$$\hat{H}^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \,\middle|\, (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n) \right\}$$

Is an Hilbert space with norm $\|u(x)\|_{H^s} := \|(1+|\xi|^2)^{s/2}\hat{u}(\xi)\|_{L^2}$

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$$u \in H^s(\Omega) \text{ if } u = U|_{\Omega} \text{ with } U \in H^s(\mathbb{R}^n)$$

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Proposition

Let Ω be a bounded $C^{0,1}$ domain; then, for any $0 < s < s_1$ we have

$$H^{s_1}(\Omega) \hookrightarrow H^s(\Omega)$$

with dense and compact embedding.

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Let $s \in (0,1)$; for any $u \in H^s(\mathbb{R}^n)$

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s} \mathscr{F} u) \qquad \forall \xi \in \mathbb{R}^n$$

Proposition

Let u, v be two $H^s(\mathbb{R}^n)$ function vanishing outside Ω; then

$$\int_\Omega v(-\Delta)^s u dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx = \int_\Omega u(-\Delta)^s v dx$$

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Fractional wave equation

$$\begin{cases} u_{tt} + (-\Delta)^{s+1} u = 0 & in \;\; \Omega \times [0, T] := Q \\ u \equiv 0 & on \;\; \partial\Omega \times [0, T] := \Sigma \\ u(x,0) = u_0(x) & in \;\; \Omega \\ u_t(x,0) = u_1(x) & in \;\; \Omega \end{cases}$$

In order to gurantee uniform velocity of propagation we need the exponent of the Fractional laplacian to be greater than 1.

Higher order Fractional laplacian

$$(-\Delta)^{s+1} := (-\Delta)(-\Delta)^s = (-\Delta)^s(-\Delta)$$

$$\mathcal{D}((-\Delta)^{s+1}) = H_0^1(\Omega) \times H^{2(s+1)}(\Omega)$$

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Energy

$$E(t) := \frac{1}{2}\int_\Omega \left\{ (\nabla u_t)^2 + \left((-\Delta)^{\frac{s+2}{2}} u \right)^2 \right\} dx$$

$$\frac{dE(t)}{dt} = 0 \Rightarrow E(t) \equiv E(0) \equiv E_0 \quad \forall t \geq 0$$

Pohozaev-type identity

$$\int_\Omega (-\Delta)^s u (x \cdot \nabla u) dx = \int_\Omega u(-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma$$

X. ROS-OTON and J. SERRA - The Pohozaev identity for the Fractional laplacian

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Proposition

For any solution of the fractional wave equation it holds

$$\begin{aligned}\frac{\Gamma(1+s)^2}{2}\int_\Sigma\left(\frac{-\Delta u}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma dt &= \frac{2s-n}{2}\int_Q\left((-\Delta)^{\frac{s+2}{2}}u\right)^2 dxdt\\ &+ \frac{2+n}{2}\int_Q(\nabla u_t)^2 dxdt\\ &+ \int_\Omega u_t(x\cdot\nabla(-\Delta u))dx\Big|_0^T\end{aligned}$$

Proof.

The identity is obtained by multiplying the equation by $x\cdot\nabla(-\Delta u)$ and integrating by parts over Q by using the Pohozaev identity. □

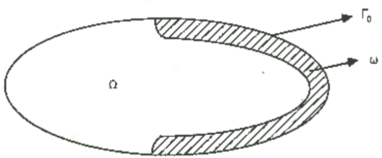
Energy estimate

Theorem

Let $E(t)$ defined as before; then, there exists two non negative constants A_1 and A_2, depending only on s, T and Ω, such that

$$A_1 E_0 \leq \int_\Sigma \left(\frac{-\Delta u}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq A_2 E_0$$

Control from a neighborhood of the boundary



We will use Hilbert Uniqueness Method

Observability inequality

$$E_0 \leq C_1 \int_0^T \int_\omega \left\{ (\nabla u_t)^2 + \left((-\Delta)^{\frac{s+2}{2}} u \right)^2 \right\} dxdt$$

By using equipartition of the energy

$$E_0 = \left\|(-\Delta)^{\frac{s+2}{2}} u_0\right\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)} \le C_2 \int_0^T \int_\omega (\nabla u_t)^2 dxdt \qquad (1)$$

Take $\phi(x,t) := \int_0^t u(x,s)ds - (-\Delta)^{-(s+1)} u_1(x)$; this ϕ satisfies the fractional wave equation with initial data

$$\phi_0 := \phi(0) = -(-\Delta)^{-(s+1)} u_1 \qquad \phi_1 := \phi_t(0) = u_0$$

Thus $(u_0, u_1) \in L^2(\Omega) \times H^{-(s+1)}(\Omega) \Rightarrow (\phi_0, \phi_1) \in H^{s+1}(\Omega) \times L^2(\Omega)$

Considering (1) for the function ϕ we have

$$\|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{H^{-s}(\Omega)} \le C_2 \int_0^T \int_\omega (\nabla u)^2 dxdt \qquad (2)$$

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RETROGRADE SYSTEM

$$\begin{cases} \psi_{tt} + (-\Delta)^{s+1} \psi = h(u) \\ \psi|_\Sigma \equiv 0 \\ \psi(x,T) = \psi'(x,T) = 0 \end{cases}$$

The solution of the retrograde system is defined by transposition: the function ψ is a solution of the problem if and only if for any solution θ of

$$\begin{cases} \theta_{tt} + (-\Delta)^{s+1} \theta = f \\ \theta|_\Sigma \equiv 0 \\ \theta(x,0) = \theta_0 \\ \theta_t(x,0) = \theta_1 \end{cases}$$

$$\int_Q \psi f dxdt - \int_\Omega (\psi_t(0)\theta_0 - \psi(0)\theta_1) dx = \int_Q \theta h dxdt \qquad (3)$$

We define the operator

$$\Lambda(u_0, u_1) := (\psi_t(x, 0), -\psi(x, 0))$$

by considering (3) with $\theta = u$ and choosing the control function

$$h(u) = div(\rho(t)u)\chi_{\{\omega\times(0,T)\}}$$

we obtain

$$\langle\Lambda(u_0, u_1); (u_0, u_1)\rangle = \int_0^T \int_\omega |\nabla u|^2 \, dxdt.$$

Thus, thanks to the observability inequality, for any given couple of initial data $(\psi_0, \psi_1) \in H^{(s+1)}(\Omega) \times L^2(\Omega)$ there exists a unique solution $(u_0, u_1) \in L^2(\Omega) \times H^{-(s+1)}(\Omega)$ of the problem $\Lambda(u_0, u_1) = (\psi_1, -\psi_0)$ and we have found a control

$$h \in L^2(0, T; H^{-1}(\omega))$$

that drives the system in rest in time T.

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THANKS FOR YOUR ATTENTION!