Eigenvalues of harmonic and poly-harmonic operators subject to mass density perturbations

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Poly-harmonic operators with Dirichlet and itermediate boundary conditions

$$\begin{cases} (-\Delta)^n u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, & \text{on } \partial\Omega\,, \\ B_m(x;D)u = B_{m+1}(x;D)u = \dots = B_n(x;D)u = 0, & \text{on } \partial\Omega\,, \end{cases}$$

$\frac{n}{2} \leq m \leq n$ if n even, $\frac{n+1}{2} \leq m \leq n$ if n odd.

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Ω is a domain in $\mathbb{R}^N$ of finite measure.

$\rho \in \mathcal{R} := \{f \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega f(x) > 0\}$

Poly-harmonic operators with Dirichlet and itermediate boundary conditions

$m = n$ Dirichlet boundary conditions:

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Cases of interest

1. Case n=1: Laplacian with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,. \end{cases}$$

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2. Case n=2:
   - Bilaplacian with Dirichlet boundary conditions

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The eigenvalue problem

The considered eigenvalue problems have a sequence of eigenvalues

$$0 < \lambda_1^{n,m}[\rho] \leq \lambda_2^{n,m}[\rho] \leq \cdots \leq \lambda_j^{n,m}[\rho] \leq \cdots$$

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Our aim is to study the dependence

$$\rho \mapsto \lambda_j^{n,m}[\rho]$$

Analyticity of the eigenvalues

Theorem

Let Ω be a domain in $\mathbb{R}^N$ of finite measure, F a nonempty finite subset of $\mathbb{N}$ and let

$$\mathcal{R}^{n,m}[F] := \{\rho \in \mathcal{R} : \lambda_j^{n,m}[\rho] \neq \lambda_l^{n,m}[\rho],\ \forall j \in F, l \in \mathbb{N} \setminus F\},$$

$$\Theta^{n,m}[F] := \{\rho \in \mathcal{R}^{n,m}[F] : \lambda_{j_1}^{n,m}[\rho] = \lambda_{j_2}^{n,m}[\rho],\ \forall j_1, j_2 \in F\}.$$

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Then $\mathcal{R}^{n,m}[F]$ is open in $L^\infty(\Omega)$ and the symmetric functions of the eigenvalues

$$\Lambda_{F,h}^{n,m}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F \\ j_1 < \dots < j_h}} \lambda_{j_1}^{n,m}[\rho] \cdots \lambda_{j_h}^{n,m}[\rho]\,, \quad h = 1, \dots, |F|$$

are analytic in $\mathcal{R}^{n,m}[F]$.

Derivatives of the eigenvalues

Theorem

Let Ω be a domain in $\mathbb{R}^N$ of finite measure, F a nonempty finite subset of $\mathbb{N}$. If $\rho \in \Theta^{n,m}[F]$ and the eigenvalues $\lambda_j^{n,m}[\rho]$ assume the common value $\lambda_F^{n,m}[\rho]$ for all $j \in F$, then the differential of $\Lambda_{F,h}^{n,m}$ at ρ is given by the formula

$$d\Lambda_{F,h}^{n,m}[\rho][\dot{\rho}] = -\left(\lambda_F^{n,m}[\rho]\right)^{h+1} \binom{|F|-1}{h-1} \sum_{l \in F} \int_{\Omega} (u_l^{n,m})^2 \dot{\rho}\, dx\,,$$

for all $\dot{\rho} \in L^\infty(\Omega)$, where $\{u_l^{n,m}\}$ is an orthonormal basis for $\lambda_F^{n,m}[\rho]$ in $H^n(\Omega) \cap H_0^m(\Omega)$.

Mass constraint

Let $M \in]0, +\infty[$. We set

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Let $U \subset L^\infty(\Omega)$ be an open set, $f : U \to \mathbb{R}$ a differentiable function.

We say that ρ is a critical mass density for f under the constraint $M[\rho] = \int_\Omega \rho dx = \mathrm{cost}$ if

$$\mathrm{Ker} dM[\rho] \subseteq \mathrm{Ker} df[\rho].$$

Critical mass densities

Theorem

Let Ω be a domain in $\mathbb{R}^N$ of finite measure, F a nonempty finite subset of $\mathbb{N}$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in \mathcal{R}^{n,m}[F]$ to $\Lambda^{n,m}_{F,h}$ has no critical mass densities in $\mathcal{R}^{n,m}[F] \cap L_M$.

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$$\sum_{i \in F} c_i u_i^2 = \mathrm{const} \Longrightarrow u_1 = \dots = u_{|F|} = 0$$

Critical mass densities

Case $n = 1, N = 1$. Let $0 < \alpha < \beta < +\infty$ and let ρ be such that

$$\alpha \leq \rho \leq \beta.$$

Critical mass densities for ρ must satisfy

$$(\rho(x) - \alpha)(\rho(x) - \beta) = 0 \quad \text{a.e. in } \Omega.$$

This result has been extended by Cox in the case $n = 1, N > 1$ for $\lambda_1[\rho]$.

Critical mass densities

Proposition

Let Ω be a domain in $\mathbb{R}^N$ of finite measure, $C \subset L^\infty(\Omega)$ a weakly compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho \in C} \operatorname{ess\,inf}_{x \in \Omega} \rho(x) > 0$. Then the functions which take $\rho \in C$ to $\lambda_j^{n,m}[\rho]$ are continuous for the weak* topology of $L^\infty(\Omega)$.*

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Theorem

Let Ω be a domain in $\mathbb{R}^N$ of finite measure, F a nonempty finite subset of $\mathbb{N}$ and $M > 0$. Let $C \subseteq \mathcal{R}^{n,m}[F]$ be a weakly compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho \in C} \operatorname{ess\,inf}_{x \in \Omega} \rho(x) > 0$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}^{n,m}$ has maxima and minima, and such points belong to $\partial C \cap L_M$.*

Mixed boundary conditions

Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$.

We consider

$$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2, \;\; \Gamma_1 \cap \Gamma_2 = \emptyset,$$

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and the class of problems

$$\begin{cases} (-\Delta)^n u = \lambda\rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, & \text{on } \Gamma_1\,, \\ B_m(x;D)u = B_{m+1}(x;D)u = \dots = B_n(x;D)u = 0, & \text{on } \Gamma_1\,, \\ N_1(x;D)u = N_2(x;D)u = \dots = N_n(x;D) = 0, & \text{on } \Gamma_2\,, \end{cases}$$

with $\frac{n}{2} \leq m \leq n$ if n even, $\frac{n+1}{2} \leq m \leq n$ if n odd.

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Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, F a nonempty finite subset of $\mathbb{N}$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in \mathcal{R}^{n,m}[F]$ to $\Lambda_{F,h}^{n,m}$ has no critical mass densities in $\mathcal{R}^{n,m}[F] \cap L_M$.

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Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, F a nonempty finite subset of $\mathbb{N}$, $M > 0$. Let $C \subseteq \mathcal{R}^{n,m}[F]$ be a weakly compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho \in C} \operatorname{ess\,inf}_{x \in \Omega} \rho(x) > 0$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}^{n,m}$ has points of maximum and minimum, and such points belong to $\partial C \cap L_M$.*

Neumann boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1.

The eigenvalue problem for the laplacian with Neumann boundary conditions is

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

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We have a sequence

$$0 < \lambda_1[\rho] \leq \lambda_2[\rho] \leq \cdots \leq \lambda_j[\rho] \leq \cdots$$

Analyticity of eigenvalues and derivatives

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, F a nonempty finite subset of $\mathbb{N}$. Then the symmetric functions of the eigenvalues $\Lambda_{F,h}$ are analytic in $\mathcal{R}[F]$.

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Theorem

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Moreover, if $\rho \in \Theta[F]$ and the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_F[\rho]$ per for all $j \in F$, then the differential of $\Lambda_{F,h}$ in ρ is given by the formula

$$d\Lambda_{F,h}[\rho][\dot{\rho}] = -(\lambda_F[\rho])^{h+1}\binom{|F|-1}{h-1}\sum_{l\in F}\int_\Omega u_l^2\,\dot{\rho}\,dx\,,$$

for all $\dot{\rho} \in L^\infty(\Omega)$, where $\{u_l\}$ is a hortonormal basis for $\lambda_F[\rho]$ in $H^{1,0}_\rho(\Omega) := \left\{u \in H^1(\Omega) : \int_\Omega u\rho dx = 0\right\}$.

Critical mass densities

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Then simple eigenvalues have no critical mass densities under the only constraint of fixed mass.

$$u = \mathrm{const} \Longrightarrow \lambda = 0$$

Critical mass densities

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $F = \{m, n\}$, with $m, n \in \mathbb{N}$, $m \neq n$. Let $\tilde{\rho} \in \mathcal{R}[F]$ continuous, such that the solutions of (1) be classic solutions and moreover their nodal domains are stokians. Then for $h = 1, 2$, $\tilde{\rho}$ is not a critical mass density for the function which takes $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$.

$$\sum_{i \in F} c_i u_i^2 = \mathrm{const}$$

Critical mass densities

Theorem

Let $\Omega \subset \mathbb{R}^N$ and F be as in Theorem 9. Let $C \subseteq \mathcal{R}[F]$ be a weakly compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C} \operatorname{ess\,inf}_\Omega \rho > 0$. Let $M > 0$ and $L_M = \{\rho \in L^\infty(\Omega) : \int_\Omega \rho = M\}$. Then for $h = 1, 2$, the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ admits points of maximum and minimum, and if for such points the solutions of problem (1) are classic solution, they belong to $\partial C \cap L_M$.*

Steklov boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1.

The eigenvalue problem for the laplacian with Steklov boundary condition is

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

$\rho \in \mathcal{R}' := \{f \in L^\infty(\partial\Omega) : \operatorname{ess\,inf}_{\partial\Omega} f(x) > 0\}$.

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We have a sequence

$$0 < \lambda_1[\rho] \le \lambda_2[\rho] \le \cdots \le \lambda_j[\rho] \le \cdots$$

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Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F a nonempty finite subset of $\mathbb{N}$. Then the symmetric functions of eigenvalues $\Lambda_{F,h}$ are analytic in $\mathcal{R}[F]$.
Moreover, if $\rho \in \Theta[F]$ and the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_F[\rho]$ for all $j \in F$, then the differential of $\Lambda_{F,h}$ at ρ is given by the formula

$$d\Lambda_{F,h}[\rho][\dot{\rho}] = -\left(\lambda_F[\rho]\right)^{h+1} \binom{|F|-1}{h-1} \sum_{l \in F} \int_{\partial\Omega} u_l^2 \dot{\rho} \, d\sigma \, ,$$

for all $\dot{\rho} \in L^\infty(\partial\Omega)$, where $\{u_l\}$ is a hortonormal basis for $\lambda_F[\rho]$ in $H^{1,0}_\rho(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u\rho d\sigma = 0\right\}$.

Critical mass densities

Proposition

Let $B = B^N(0,1)$ be the unit ball in $\mathbb{R}^N$, S_N the $(N-1)$-dimensional measure of ∂B, $F = \{1, ..., N\}$, $M > 0$. Then the constant mass density $\rho_M = \frac{M}{S_N}$ is a critical mass density for $\Lambda_{F,h}$ for $h = 1, ..., N$ under the constraint $\int_{\partial\Omega} \rho\sigma = M$.

Spectral convergence

Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $M > 0$, ω_N the volume of B, S_N the $(N-1)$-dimensional measure of ∂B. Let B_n be the ball $B(0, 1-\frac{1}{n})$. Let $\rho_n \in \mathcal{R}$ be defined by

$$\rho_n(x) := \begin{cases} \frac{1}{n}, & \text{if } x \in B_n, \\ \frac{M - \frac{\omega_N}{n}\left(1-\frac{1}{n}\right)^N}{\omega_N\left(1-\left(1-\frac{1}{n}\right)^N\right)}, & \text{if } x \in B \setminus B_n, \end{cases} \tag{3}$$

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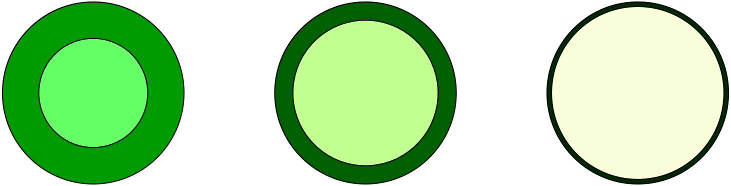


Spectral convergence

Theorem

Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $M > 0$, S_N the $(N-1)$-dimensional measure of ∂B and $\rho_n \in \mathcal{R}$ be defined as in (3). Let $\lambda_j[\rho_n]$ be the eigenvalues of problem (1) on B for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$ be the eigenvalues of (2) on B corresponding to the constant density $\frac{M}{S_N}$. Then for all $j \in \mathbb{N}$ we have $\lim_{n\to+\infty} \lambda_j[\rho_n] = \overline{\lambda}_j$.

Spectral convergence

Theorem

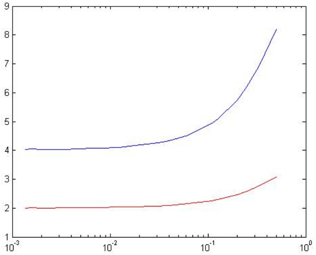
Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2, $M > 0$. We denote by Ω_n the set $\left\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \frac{1}{n}\right\}$. Let $\rho_n \in \mathcal{R}$ be defined by

$$\rho_n(x) := \begin{cases} \frac{1}{n}, & \text{if } x \in \Omega_n, \\ \frac{M - \frac{1}{n}|\Omega_n|}{|\Omega \setminus \Omega_n|}, & \text{if } x \in \Omega \setminus \Omega_n, \end{cases}$$

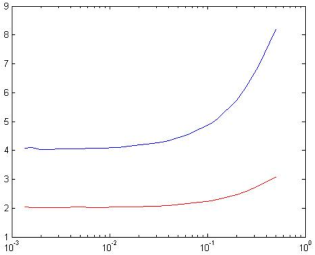
Let $\lambda_j[\rho_n]$ be the eigenvalues of problem (1) for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$ be the eigenvalues of problem (2) corresponding to the constant mass density $\frac{M}{|\partial\Omega|}$. Then for all $j \in \mathbb{N}$ we have $\lim_{n\to+\infty} \lambda_j[\rho_n] = \overline{\lambda}_j$.

Spectral convergence

Numerical experiments on $B(0,1)$ in $\mathbb{R}^2$ and $M = \pi$. The first(double) and second (double, blue) eigenvalues for the Steklov problem with constant surface density $\rho_\pi \equiv \frac{1}{2}$ on ∂B are $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 4$.



(a) λ_1, λ_3



(b) λ_2, λ_4

Further remarks

1. The above problems can be considered for a more general class of differential elliptic operators

$$\mathcal{L}u = \sum_{0 \le |\alpha|, |\beta| \le n} (-1)^{|\alpha|} D^{\alpha}(A_{\alpha\beta} D^{\beta} u),$$

subject to homogeneous b.c. on a domain Ω in $\mathbb{R}^N$ of finite measure. $A_{\alpha\beta}$ are fixed bounded real-valued functions s.t. $A_{\alpha\beta} = A_{\beta\alpha}$.

2. We can consider the weak formulation of the problem on a closed subspace $V(\Omega)$ of $H^n(\Omega)$ s.t. the inclusion $V(\Omega) \subset L^2(\Omega)$ is compact.
E.g., we can consider the problem for poly-harmonic operators with intermediate boundary conditions in the space $H^n(\Omega) \cup H^m_0(\Omega)$ for all $0 \le m \le n$.

Further remarks

1. In the general setting we must assume that $A_{\alpha\beta}$, $V(\Omega)$ are s.t. the Gårding's inequality holds, i.e., we assume that there exist $a, b > 0$ s.t.
$$a\|u\|^2_{H^n(\Omega)} \leq \mathcal{Q}[u,u] + b\|u\|^2_{L^2(\Omega)}\,,$$
for all $u \in V(\Omega)$.
2. In the proofs of weak* continuity of eigenvalues it is enough to require that the set C is bounded.
3. In the general setting, the result of non-existence of critical points for symmetric functions of eigenvalues holds when $V(\Omega) \subset H^1_{0,\Gamma}(\Omega)$. The statement for simple eigenvalues is a general fact instead.

Ref. Lamberti P.D., Provenzano L., *A maximum principle in spectral optimization problems for elliptic operators subject to mass density perturbations*. arXiv:1205.5624

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Bilaplacian with Neumann conditions

1. n=2, N=2

$$\begin{cases} (-\Delta)^2 u = \lambda\rho u, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial\nu^2} = 0, & \text{on } \partial\Omega\,, \\ \frac{d}{ds}\frac{\partial^2 u}{\partial\nu\partial t} + \frac{\partial\Delta u}{\partial\nu} = 0, & \text{on } \partial\Omega\,; \end{cases}$$

2. n=2, N>2

$$\begin{cases} (-\Delta)^2 u = \lambda\rho u, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial\nu^2} = 0, & \text{on } \partial\Omega\,, \\ \mathrm{div}_{\partial\Omega}\left(\mathrm{P}_{\partial\Omega}\left[(D^2 u).\nu\right]\right) + \frac{\partial\Delta u}{\partial\nu} = 0, & \text{on } \partial\Omega\,; \end{cases}$$

A bounded open set Ω in $\mathbb{R}^N$ is called stokian if its regular boundary $\partial_{\rm reg}\Omega$ has finite $(N-1)$ dimensional measure and $\partial\Omega \setminus \partial_{\rm reg}\Omega$ has zero $(N-1)$ dimensional measure.