Inertial Manifolds for Dissipative PDEs

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Introduction

Characterize the longtime behavior of dissipative PDEs in infinite dimensional Hilbert spaces H, e.g.,

$$\frac{d}{dt}u + Au = F(u)$$
$$u|_{t=0} = u_0, \qquad\qquad u_0 \in H,$$

where A is linear, positive, self-adjoint and A^{-1} is compact.

Assuming global well-posedness
$\Longrightarrow$ generates a semigroup in H, $S(t) : H \to H$, $t \geq 0$,

$$\begin{aligned} S(t) \circ S(s) &= S(t+s) \qquad t, s \geq 0 \\ S(0) &= Id \\ (t, x) &\mapsto S(t)x \qquad \text{continuous} \end{aligned}$$

Longtime Dynamics

Many dissipative systems posses a global attractor $\mathcal{A}$

(a) $\emptyset \neq \mathcal{A} \subset X$ compact,

(b) $S(t)\mathcal{A} = \mathcal{A}$, $t \geq 0$, invariant,

(c) attracts every bounded set $D \subset X$,

$$\lim_{t\to\infty} \mathrm{dist}_H(S(t)D, \mathcal{A}) = 0.$$

Properties:

$\mathcal{A}$ unique, minimal closed set attracting all bounded sets, in most cases, of finite fractal dimension

Aim:

Describe dynamics on $\mathcal{A}$

Motivation

Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a basis of H of eigenfunctions of A,

$$u(t) = \sum_{j\in\mathbb{N}} \langle u(t), \varphi_j \rangle \varphi_j = \sum_{j\in\mathbb{N}} c_j(t)\varphi_j,$$

We assume that for large n

$$c_j(t) = \Phi_j(c_1(t), \dots, c_n(t)) + \text{error}, \qquad j > n,$$

and the error decays exponentially.

We write $H = P_n H \oplus Q_n H$ and

$$u(t) = \sum_{j=1}^{n} c_j(t)\varphi_j + \sum_{j>n} c_j(t)\varphi_j = p(t) + q(t).$$

Inertial System

$$q(t) = \Phi(p(t)) + \text{error}$$

The graph of Φ is an n-dimensional manifold $\mathcal{M}$,

$$\mathcal{G}(\Phi) = \{u \mid u = p + \Phi(p),\ p \in P_n H\}.$$

We require that

- $\mathcal{M}$ contains $\mathcal{A}$
- $\mathcal{M}$ is positively invariant
- and exponentially attracting.

If $u \in \mathcal{M}$, then $u = p + \Phi(p)$ and

$$\frac{d}{dt}p + Ap = P_n F(u) = P_n F(p + \Phi(p))$$

is a finite set of ODEs, called inertial system.

Inertial Manifolds

An inertial manifold $\mathcal{M}$ is a finite dimensional Lipschitz manifold, which is positively invariant and attracts all trajectories exponentially, i.e., there exists $k > 0$ such that

$$\mathrm{dist}_H(S(t)u_0, \mathcal{M}) \leq c(|u_0|)e^{-kt} \qquad \forall u_0 \in H.$$

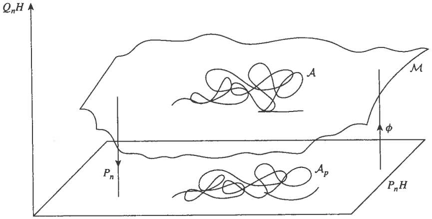

Figure 15.1. Via the inertial manifold $\mathcal{M}$, the inertial from gives rise to a set of ODEs on $P_nH \simeq \mathbb{R}^n$ with a global attractor $\mathcal{A}_p = P_n\mathcal{A}$. [1]

[1] J. C. Robinson, *"Infinite Dimensional Dynamical Systems"*. Cambridge University Press, Cambridge (2001).

Strong Squeezing Property

Let $u(t) = p(t) + q(t)$ and $\overline{u}(t) = \overline{p}(t) + \overline{q}(t)$ be two solutions.
The strong squeezing property is satisfied if

- $|q(0) - \overline{q}(0)| \leq |p(0) - \overline{p}(0)|$ implies that

$$|q(t) - \overline{q}(t)| \leq |p(t) - \overline{p}(t)| \qquad \forall\, t \geq 0,$$

- and if $|q(t) - \overline{q}(t)| > |p(t) - \overline{p}(t)|$, for some $t > 0$, then

$$|q(t) - \overline{q}(t)| \leq e^{-kt}|q(0) - \overline{q}(0)| \qquad k > 0.$$

Strong Squeezing Property

Proposition

If the strong squeezing property holds, then there exists a Lipschitz function $\Phi : P_n H \to Q_n H$,

$$|\Phi(p) - \Phi(\overline{p})| \leq |p - \overline{p}| \qquad \forall\, p, \overline{p} \in P_n H$$

such that $\mathcal{A} \subset \mathcal{G}(\Phi)$.

Existence of Inertial Manifolds

An equation is appropriately prepared if

- there exists an absorbing ball B in H such that $B \cup P_n H$ is positively invariant and
- $P_n S(t)\,(P_n H) = P_n H \qquad \forall\, t \geq 0.$

Theorem

If an equation is appropriately prepared and the strong squeezing property holds, then there exists an inertial manifold $\mathcal{M}$, given as the graph of a Lipschitz function $\Phi : P_n H \to Q_n H$.

Spectral Gap Condition

We consider

$$\frac{d}{dt}u + Au = F(u),$$

where F is globally Lipschitz in H,

$$|F(u) - F(v)| \leq C|u - v| \qquad \forall u, v \in H.$$

Proposition

If there exists $n \in \mathbb{N}$ such that the eigenvalues λ_n and λ_{n+1} of A satisfy

$$\lambda_{n+1} - \lambda_n \geq 4C,$$

then, the strong squeezing property holds.

Spectral Gap Condtion: Reaction Diffusion Equations

Consider

$$\frac{\partial}{\partial t}u = \Delta u + f(u) \qquad \Omega \times (0,\infty),$$
$$u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain.

The spectral gap condition $\lambda_{n+1} - \lambda_n > 4C$ is restrictive:

- For general Ω :

$$\lambda_n \sim n^{\frac{2}{m}}$$

Concluding remarks

For the Dirichlet Laplacian the spectral gap condition is, e.g., known for

- 1–dimensional domains
- particular 2-dimensional domains

For uniformly elliptic operators of order $2l$ in bounded domains in $\mathbb{R}^m$ we have

$$\lambda_n \sim n^{\frac{2l}{m}} \qquad \text{(Weyl formula).}$$

More general: *Approximate inertial manifolds*.
The construction is not based on the spectral gap condition.

References

- J. C. Robinson, *“Infinite Dimensional Dynamical Systems”*. Cambridge University Press, Cambridge (2001).
- R. Temam, *“Infinite Dimensional Dynamical Systems in Mechanics and Physics”*. 2nd edition, Springer Verlag, New York (1997).