Hardy Inequalities for a Class of Degenerate Elliptic Operators

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Introduction

Classical Hardy inequality:

- Important tool to study linear and non-linear PDEs.
- First proved by Hardy in 1920 for one dimensional case. Optimality of the constant by Landau in 1926.
- Later, generalized for the N-dimensional case.
- Over the years, improved and extended in various directions.

Aim:

- Hardy type in equalities for a class of degenerate elliptic operators.

N-dimensional Hardy Inequality

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a domain, $p > 1$ and $N > p$. Then, there exists a constant $c > 0$ such that

$$c \int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \int_\Omega |\nabla u(x)|^p dx,$$

for all $u \in W_0^{1,p}(\Omega)$.
If the origin $\{0\}$ belongs to the set Ω, the optimal constant is $c = \left(\frac{N-p}{p}\right)^p$, but not attained in $W_0^{1,p}(\Omega)$.

A Simple Approach to Hardy Inequalities [1]

For simplicity, let $p = 2$. For $u \in C_0^1(\Omega)$ and $h \in C^1(\Omega; \mathbb{R}^N)$ such that $\mathrm{div} h > 0$ the divergence theorem implies

$$\int_\Omega u^2 \,\mathrm{div} h \, dx = -2 \int_\Omega u \nabla u \cdot h \, dx.$$

Applying Hölder's inequality we obtain

$$\int_\Omega u^2 \,\mathrm{div} h \, dx \le 2 \left(\int_\Omega u^2 \,\mathrm{div} h \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 \frac{h^2}{\mathrm{div} h} \, dx \right)^{\frac{1}{2}}.$$

Choosing the vector field $h(x) = \frac{x}{|x|^2 + \epsilon}$ and taking the limit $\varepsilon \to 0$ leads to Hardy's inequality.

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[1] E. Mitidieri, *A simple approach to Hardy inequalities,* Mat. Zametki 67, 189–220 (2000).

Δ_λ-Laplacians [2]: $\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \cdots + \lambda_k^2 \Delta_{x^{(k)}}$

We write $(x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$ and assume

$$\lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{\alpha_{ij}}, \quad \alpha_{ij} \geq 0, \ \text{where } \alpha_{ij} = 0 \text{ for } i \leq j.$$

Then, there exists a *group of dilations* $(\delta_r)_{r>0}$,

$$\delta_r(x^{(1)}, \ldots, x^{(k)}) = (r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)}), \quad 1 = \sigma_1 \leq \sigma_i,$$

such that λ_i is δ_r-*homogeneous of degree* $\sigma_i - 1$,

$$\lambda_i(\delta_r(x)) = r^{\sigma_i - 1} \lambda_i(x) \qquad \forall x \in \mathbb{R}^N, \ r > 0,$$

and the *homogeneous dimension* of $\mathbb{R}^N$ w.r.t. $(\delta_r)_{r>0}$ is

$$Q := \sigma_1 N_1 + \cdots + \sigma_k N_k.$$

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[2] A.E. Kogoj, E. Lanconelli, *On semilinear Δ_λ-Laplace equation,* Nonlinear Anal. 75, 4637–4649 (2012).

Examples

1. Grushin type operators:

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y$$

where $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $\alpha \geq 0$. We find

$$\delta_r(x, y) = \left(rx, r^{\alpha+1}y\right),$$

and $Q = N_1 + (\alpha + 1)N_2$.

Remark: Hardy inequalities for Grushin type operators obtained by D'Ambrosio [3].

[3] L. D'Ambrosio, *Hardy inequalities related to Grushin type operators*, Proc. Amer. Math. Soc. 132, 725–734 (2003).

Examples

2. Let $\alpha, \beta, \gamma \geq 0$. For the operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, we find

$$\delta_r(x, y, z) = \left(rx, r^{\alpha+1}y, r^{\beta+(\alpha+1)\gamma+1}z\right),$$

and $Q = N_1 + (\alpha+1)N_2 + (\beta + (\alpha+1)\gamma + 1)N_3$.

Hardy Inequalities for Δ_λ-Laplacians

We denote by $\mathring{W}^{1,p}_\lambda(\Omega)$ the closure of $C^1_0(\Omega)$ with respect to

$$\|u\|_{\mathring{W}^{1,p}_\lambda(\Omega)} = \left(\int_\Omega |\nabla_\lambda u|^p dx \right)^{\frac{1}{p}},$$

where $\nabla_\lambda = (\lambda_1 \nabla_{x^{(1)}}, \ldots, \lambda_k \nabla_{x^{(k)}})$.
We obtain a large family of Hardy type inequalities of the form

$$\left(\frac{Q-c}{p} \right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]^p_\lambda} \varphi(x)\, dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \, \psi(x)\, dx.$$

where φ and ψ are certain weight functions, and $[[\cdot]]_\lambda$ is a homogeneous weighted norm.

Idea of the Proof

The proof is based on the following lemma.

Lemma

Let $\varepsilon > 0$ and $h \in C^1(\Omega; \mathbb{R}^N)$ be such that $\mathrm{div}_{\lambda^\varepsilon} h > 0$. Then, for every $p > 1$ and $u \in C_0^1(\Omega)$ we have

$$\int_\Omega |u(x)|^p \mathrm{div}_{\lambda^\varepsilon} h(x)\, dx \le p^p \int_\Omega \frac{|h(x)|^p}{(\mathrm{div}_{\lambda^\varepsilon} h(x))^{p-1}} |\nabla_{\lambda^\varepsilon} u(x)|^p\, dx,$$

where $\lambda_i^\varepsilon(x) := \prod_{j=1}^k \left(|x^{(j)}|^2 + \varepsilon\right)^{\frac{\alpha_{ji}}{2}}, \;\; i = 1, \ldots, k.$

We first consider Hardy type inequalities of the form

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]_\lambda^p}\, dx \leq \int_\Omega \psi(x)|\nabla_\lambda u(x)|^p\, dx.$$

Motivated by the lemma we look for a function h satisfying

$$\operatorname{div}_\lambda h(x) = \frac{Q-p}{[[x]]_\lambda^p}.$$

If we choose

$$h(x) = \frac{1}{[[x]]_\lambda^p}\left(\frac{\sigma_1 x^{(1)}}{\lambda_1(x)}, \ldots, \frac{\sigma_k x^{(k)}}{\lambda_k(x)}\right), \qquad \text{then}$$

$$\operatorname{div}_\lambda h(x) = \frac{Q}{[[x]]_\lambda^p} - p\frac{1}{[[x]]_\lambda^{p+1}} \sum_{i=1}^{k} \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}([[x]]_\lambda).$$

Consequently, $[[\cdot]]_\lambda$ should fulfill

$$\sum_{i=1}^{k} \sigma_i x^{(i)} \cdot \nabla_{x^{(i)}}([[x]]_\lambda) = [[x]]_\lambda.$$

On the other hand, we obtain

$$|h(x)|^2 = \frac{\prod_{i=1}^{k} \lambda_i(x)^{-2}}{[[x]]_\lambda^{2p}} \left(\prod_{j\neq 1} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j\neq k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right),$$

which motivates to consider

$$[[x]]_\lambda = \left(\prod_{j\neq 1} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j\neq k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}}.$$

Hardy Inequalities for Δ_λ-Laplacians

One particular case:

Theorem

Let $p > 1$ and $Q > p$. Then,

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{\prod_{i=1}^k \lambda_i(x)^p}{[[x]]_\lambda^{p(1+\sum_{i=1}^k(\sigma_i-1))}} |u(x)|^p \, dx \le \int_\Omega |\nabla_\lambda u(x)|^p \, dx$$

for every $u \in \mathring{W}_\lambda^{1,p}(\Omega)$, where

$$[[x]]_\lambda = \left(\prod_{i\neq 1} \lambda_i(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} \lambda_i(x)^2 \sigma_k^2 |x^{(k)}|^2\right)^{\frac{1}{2(1+\sum_{i=1}^k(\sigma_i-1))}}$$

Proof

We consider the function

$$h(x) = \frac{\prod_{i=1}^{k} \lambda_i^{\varepsilon}(x)^p}{[[x]]_{\varepsilon,\lambda}^{p(1+\sum_{i=1}^{k}(\sigma_i - 1))}} \left(\frac{\sigma_1 x^{(1)}}{\lambda_1^{\varepsilon}(x)}, \dots, \frac{\sigma_k x^{(k)}}{\lambda_k^{\varepsilon}(x)} \right),$$

where

$$[[x]]_{\varepsilon,\lambda} = \left(\sum_{j=1}^{k} \Big(\prod_{i \neq j} \lambda_i^{\varepsilon}(x)^2 \sigma_j^2 (|x^{(j)}|^2 + \varepsilon) \Big) \right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\sigma_i - 1))}},$$

apply the lemma and pass to the limit $\varepsilon \to 0$ in the resulting inequality.

Remarks

- For the particular case of Grushin type operators we recover previous inequalities by D'Ambrosio, which were proved to be sharp.
- The fundamental solution for Grushin type operators, as well as the function that "optimizes" the Hardy inequality, are known.
- We obtain explicit constants in our inequalities, but are currently unable to show its optimality.
  For general Δ_λ-Laplacians the fundamental solution and the functions that "optimize" our inequalities are unknown.

References

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