Hardy Inequalities for a Class of Degenerate Elliptic Operators

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February 17, 2014
Introduction

Classical Hardy inequality:

- Important tool to study linear and non-linear PDEs.
- First proved by Hardy in 1920 for one dimensional case. Optimality of the constant by Landau in 1926.
- Later, generalized for the $N$-dimensional case.
- Over the years, improved and extended in various directions.

Aim:

- Hardy type in equalities for a class of degenerate elliptic operators.
**N-dimensional Hardy Inequality**

**Theorem**

Let $\Omega \subset \mathbb{R}^N$ be a domain, $p > 1$ and $N > p$. Then, there exists a constant $c > 0$ such that

$$c \int_{\Omega} \frac{|u(x)|^p}{|x|^p} \, dx \leq \int_{\Omega} |\nabla u(x)|^p \, dx,$$

for all $u \in W^{1,p}_0(\Omega)$.

If the origin $\{0\}$ belongs to the set $\Omega$, the optimal constant is $c = \left(\frac{N-p}{p}\right)^p$, but not attained in $W^{1,p}_0(\Omega)$. 

Hardy Inequalities for a Class of Degenerate Elliptic Operators
A Simple Approach to Hardy Inequalities $^1$

For simplicity, let $p = 2$. For $u \in C^1_0(\Omega)$ and $h \in C^1(\Omega; \mathbb{R}^N)$ such that $\text{div} h > 0$ the divergence theorem implies

$$\int_{\Omega} u^2 \text{div} h \, dx = -2 \int_{\Omega} u \nabla u \cdot h \, dx.$$

Applying Hölder’s inequality we obtain

$$\int_{\Omega} u^2 \text{div} h \, dx \leq 2 \left( \int_{\Omega} u^2 \text{div} h \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla u|^2 \frac{h^2}{\text{div} h} \, dx \right)^{\frac{1}{2}}.$$

Choosing the vector field $h(x) = \frac{x}{|x|^2 + \epsilon}$ and taking the limit $\epsilon \to 0$ leads to Hardy’s inequality.

\( \Delta_\lambda \)-Laplacians \(^2\): \( \Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \cdots + \lambda_k^2 \Delta_{x^{(k)}} \)

We write \((x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}\) and assume

\[
\lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{\alpha_{ij}}, \quad \alpha_{ij} \geq 0, \quad \text{where } \alpha_{ij} = 0 \text{ for } i \leq j.
\]

Then, there exists a group of dilations \((\delta_r)_{r>0}\),

\[
\delta_r(x^{(1)}, \ldots, x^{(k)}) = (r^{\sigma_1} x^{(1)}, \ldots, r^{\sigma_k} x^{(k)}), \quad 1 = \sigma_1 \leq \sigma_i,
\]

such that \(\lambda_i\) is \(\delta_r\)-homogeneous of degree \(\sigma_i - 1\),

\[
\lambda_i(\delta_r(x)) = r^{\sigma_i - 1} \lambda_i(x) \quad \forall x \in \mathbb{R}^N, \ r > 0,
\]

and the homogeneous dimension of \(\mathbb{R}^N\) w.r.t. \((\delta_r)_{r>0}\) is

\[
Q := \sigma_1 N_1 + \cdots + \sigma_k N_k.
\]

Examples

1. Grushin type operators:

\[ \Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y \]

where \((x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}\) and \(\alpha \geq 0\). We find

\[ \delta_r (x, y) = (rx, r^{\alpha+1}y) \]

and \(Q = N_1 + (\alpha + 1)N_2\).

*Remark:* Hardy inequalities for Grushin type operators obtained by D’Ambrosio \(^3\).

Examples

2. Let $\alpha, \beta, \gamma \geq 0$. For the operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha} \Delta_y + |x|^{2\beta} |y|^{2\gamma} \Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, we find

$$\delta_r (x, y, z) = \left( rx, r^{\alpha+1} y, r^{\beta+(\alpha+1)\gamma+1} z \right),$$

and $Q = N_1 + (\alpha + 1)N_2 + (\beta + (\alpha + 1)\gamma + 1)N_3$. 
Hardy Inequalities for $\Delta_\lambda$-Laplacians

We denote by $\hat{W}^1_p(\Omega)$ the closure of $C_0^1(\Omega)$ with respect to

$$\|u\|_{\hat{W}^1_p(\Omega)} = \left(\int_\Omega |\nabla_\lambda u|^p \, dx\right)^{\frac{1}{p}},$$

where $\nabla_\lambda = (\lambda_1 \nabla x^{(1)}, \ldots, \lambda_k \nabla x^{(k)})$.

We obtain a large family of Hardy type inequalities of the form

$$\left(\frac{Q - c}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{[[x]]^p_\lambda} \varphi(x) \, dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \psi(x) \, dx,$$

where $\varphi$ and $\psi$ are certain weight functions, and $[[\cdot]]_\lambda$ is a homogeneous weighted norm.
Idea of the Proof

The proof is based on the following lemma.

**Lemma**

Let \( \varepsilon > 0 \) and \( h \in C^1(\Omega; \mathbb{R}^N) \) be such that \( \text{div}_{\lambda \varepsilon} h > 0 \). Then, for every \( p > 1 \) and \( u \in C^1_0(\Omega) \) we have

\[
\int_{\Omega} |u(x)|^p \text{div}_{\lambda \varepsilon} h(x) \, dx \leq p^p \int_{\Omega} \frac{|h(x)|^p}{(\text{div}_{\lambda \varepsilon} h(x))^{p-1}} |\nabla_{\lambda \varepsilon} u(x)|^p \, dx,
\]

where \( \lambda_{\varepsilon}^j(x) := \prod_{j=1}^{k} \left( |x^{(j)}|^2 + \varepsilon \right)^{\frac{\alpha_{ji}}{2}}, \quad i = 1, \ldots, k. \)
We first consider Hardy type inequalities of the form

\[
\left( \frac{Q - p}{p} \right)^p \int_{\Omega} \frac{|u(x)|^p}{\omega(x)}\, dx \leq \int_{\Omega} \psi(x)|\nabla \lambda u(x)|^p\, dx.
\]

Motivated by the lemma we look for a function \( h \) satisfying

\[
\text{div}_\lambda h(x) = \frac{Q - p}{\omega(x)}.
\]

If we choose

\[
h(x) = \frac{1}{\omega(x)} \left( \frac{\sigma_1 x^{(1)}}{\lambda_1(x)} , \cdots , \frac{\sigma_k x^{(k)}}{\lambda_k(x)} \right),
\]

then

\[
\text{div}_\lambda h(x) = \frac{Q}{\omega(x)} - p \frac{1}{\omega(x)} \sum_{i=1}^k \sigma_i x^{(i)} \cdot \nabla x^{(i)}(\omega(x)).
\]
Consequently, $[[\cdot]]_\lambda$ should fulfill

$$
\sum_{i=1}^{k} \sigma_j x^{(i)} \cdot \nabla x^{(i)} ([ [ x ] ]_\lambda) = [ [ x ] ]_\lambda.
$$

On the other hand, we obtain

$$
|h(x)|^2 = \frac{\prod_{i=1}^{k} \lambda_i(x)^{-2}}{[ [ x ] ]_\lambda^{2p}} \left( \prod_{j \neq 1}^{k} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j \neq k}^{k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right),
$$

which motivates to consider

$$
[ [ x ] ]_\lambda = \left( \prod_{j \neq 1}^{k} \lambda_j(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{j \neq k}^{k} \lambda_j(x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{\frac{1}{2(1+\sum_{i=1}^{k} (\sigma_i-1))}}.
$$
Hardy Inequalities for $\Delta_\lambda$-Laplacians

One particular case:

**Theorem**

Let $p > 1$ and $Q > p$. Then,

$$
\left( \frac{Q - p}{p} \right)^p \int_\Omega \frac{\prod_{i=1}^k \lambda_i(x)^p}{[[x]]_{\lambda}^{p(1+\sum_{i=1}^k (\sigma_i - 1))}} |u(x)|^p \, dx \leq \int_\Omega |\nabla_\lambda u(x)|^p \, dx
$$

for every $u \in \dot{W}_\lambda^{1,p}(\Omega)$, where

$$
[[x]]_{\lambda} = \left( \prod_{i \neq 1} \lambda_i(x)^2 \sigma_1^2 |x^{(1)}|^2 + \cdots + \prod_{i \neq k} \lambda_i(x)^2 \sigma_k^2 |x^{(k)}|^2 \right)^{1 \over 2(1+\sum_{i=1}^k (\sigma_i - 1))}
$$
Proof

We consider the function

\[ h(x) = \frac{\prod_{i=1}^{k} \lambda_i^\varepsilon(x)^p}{[[x]]_{\varepsilon,\lambda}^{p(1+\sum_{i=1}^{k} (\sigma_i - 1))}} \left( \frac{\sigma_1 x^{(1)}}{\lambda_1^\varepsilon(x)}, \ldots, \frac{\sigma_k x^{(k)}}{\lambda_k^\varepsilon(x)} \right), \]

where

\[ [[x]]_{\varepsilon,\lambda} = \left( \sum_{j=1}^{k} \left( \prod_{i \neq j} \lambda_i^\varepsilon(x)^2 \sigma_j^2 (|x^{(j)}|^2 + \varepsilon) \right) \right)^{\frac{1}{2(1+\sum_{i=1}^{k} (\sigma_i - 1))}}, \]

apply the lemma and pass to the limit \( \varepsilon \to 0 \) in the resulting inequality.
Remarks

- For the particular case of Grushin type operators we recover previous inequalities by D’Ambrosio, which were proved to be sharp.
- The fundamental solution for Grushin type operators, as well as the function that "optimizes" the Hardy inequality, are known.
- We obtain explicit constants in our inequalities, but are currently unable to show its optimality. For general $\Delta_{\lambda}$-Laplacians the fundamental solution and the functions that "optimize" our inequalities are unknown.
References