Evolution problems involving the fractional Laplace operator: HUM control and Fourier analysis

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We analyse the control problem for the fractional Schrödinger equation

$$iu_t + (-\Delta)^s u = 0$$

and for the fractional wave equation

$$u_{tt} + (-\Delta)^{s+1} u = 0$$

on a bounded $C^{1,1}$ domain Ω of $\mathbb{R}^n$. We focus on the control from a neighbourhood of the boundary $\partial\Omega$.

Fractional laplacian

$$(-\Delta)^s u(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \qquad s \in (0,1)$$

$$c_{n,s} := \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2} \Gamma(1-s)}$$

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Fractional Schrödinger equation

$$\begin{cases} iu_t + (-\Delta)^s u = 0 & in \quad \Omega \times [0,T] := Q \\ u \equiv 0 & in \quad (\mathbb{R}^n \setminus \Omega) \times [0,T] \\ u(x,0) = u_0(x) & in \quad \Omega \end{cases} \tag{1}$$

Fractional wave equation

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Thanks to Hille-Yosida theorem, both problems are well posed.

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Thanks to Hille-Yosida theorem, both problems are well posed.

Internal control result - Schrödinger equation

Let Ω be a bounded $C^{1,1}$ domain of $\mathbb{R}^n$.
Let us consider the nonhomogeneous fractional Schrödinger equation

$$\begin{cases} iy_t + (-\Delta)^s y = h\chi_{\omega\times[0,T]} & \textit{in } \Omega\times[0,T] := Q \\ y \equiv 0 & \textit{on } (\mathbb{R}^n\setminus\Omega)\times[0,T] \\ y(x,0) = y_0(x) & \textit{in } \Omega \end{cases} \tag{3}$$

where ω is a neighbourhood of the boundary of the domain and χ is the characteristic function.

Theorem

Let $T > 0$ and $\omega \subset \Omega$ be a neighbourhood of the boundary of the domain. Then, for any $y_0 \in L^2(\Omega)$ there exists $h \in L^2(\omega\times[0,T])$ such that the solution of (3) satisfies $y(x,T) = 0$.

Proposition

For any solution u of

$$\begin{cases} iu_t + (-\Delta)^s u = 0 & \text{in } \Omega \times [0,T] := Q \\ u \equiv 0 & \text{on } (\mathbb{R}^n \setminus \Omega) \times [0,T] \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

it holds

$$\Gamma(1+s)^2 \int_\Sigma \left(\frac{|u|}{\delta^s}\right)^2 (x\cdot\nu)\, d\sigma\, dt = 2s \int_0^T \left\|(-\Delta)^{s/2} u\right\|^2_{L^2(\Omega)} dt + \Im \int_\Omega \bar{u}(x\cdot\nabla u)\, dx \Big|_0^T$$

$$\Sigma := \partial\Omega \times [0,T] \qquad \delta = \delta(x) := d(x,\partial\Omega)$$

Proof.

The identity is obtained by multiplying the equation by $x \cdot \nabla \bar{u} + \frac{n}{2}\bar{u}$, taking the real part and integrating over Q by using the Pohozaev identity

$$\int_\Omega (-\Delta)^s u (x \cdot \nabla u) dx = \frac{2s-n}{2} \int_\Omega u (-\Delta)^s u dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma \tag{4}$$

X. ROS-OTON and J. SERRA - The Pohozaev identity for the Fractional laplacian □

Theorem

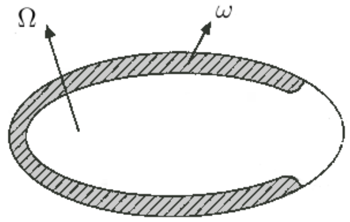
For any u solution of (1) there exist two non negative constants A_1 and A_2, depending only on n, s, T and Ω, such that

$$A_1\|u_0\|^2_{H^s(\Omega)} \leq \int_\Sigma \left(\frac{|u|}{\delta^s}\right)^2 (x\cdot\nu)d\sigma dt \leq A_2\|u_0\|^2_{H^s(\Omega)}$$

Proof.

The proof is merely technical. We use some interpolation results and some compactness-uniqueness arguments. □

Control from a neighbourhood of the boundary



We will use Hilbert Uniqueness Method

Observability inequality

$$\|u_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |u|^2 dxdt$$

Proof of the observability inequality

1. $\|u_0\|^2_{H^s(\Omega)} \leq C_1 \int_0^T \int_{\hat{\omega}} |\nabla u|^2 dxdt \qquad (\Omega \cap \overline{\hat{\omega}}) \subset \omega$

2. $\|u_0\|^2_{H^s(\Omega)} \leq C_2 \int_0^T \left(\|u_t\|_{H^{-s}(\omega)} + \|u_0\|_{L^2(\omega)} \right) dt$

   $\Rightarrow \|u_0\|^2_{H^s(\Omega)} \leq C_3 \int_0^T \|u_t\|^2_{H^{-s}(\omega)} dt$

3. $\|u_0\|^2_{H^{-s}(\Omega)} \leq C_4 \int_0^T \|u\|^2_{H^{-s}(\omega)} dt$

4. $\|u_0\|^2_{H^s(\Omega)} \leq C_5 \|u\|^2_{L^2(0,T;H^s(\omega))}, \; \|u_0\|^2_{H^{-s}(\Omega)} \leq C_5 \|u\|^2_{L^2(0,T;H^{-s}(\omega))}$

5. The proof is concluded by interpolation.

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Control result via HUM

$$\begin{cases} iu_t + (-\Delta)^s u = 0 \\ u|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ u(x,0) = u_0(x) \end{cases}$$

BACKWARD SYSTEM

$$\begin{cases} iy_t + (-\Delta)^s y = u\chi_{\omega \times [0,T]} \\ y|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ y(x,T) = 0 \end{cases}$$

The solution of the backward system is defined by transposition with a solution θ of

$$\begin{cases} i\theta_t + (-\Delta)^s \theta = f \\ \theta|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ \theta(x,0) = \theta_0 \end{cases}$$

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We consider the operator $\Lambda : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\Lambda u_0 := -iy(x,0).$$

It is immediate to check the identity

$$\langle \Lambda u_0; u_0 \rangle = \int_0^T \int_\omega |u|^2 dxdt$$

Thus, thanks to the observability inequality, Λ is an isomorphism from $L^2(\Omega)$ to $L^2(\Omega)$. Hence, given $y_0 \in L^2(\Omega)$ we can choose the control function

$$h := u\big|_\omega$$

where u is the solution of (1) with intial datum $u_0 = \Lambda^{-1}(-iy_0)$.

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Fractional wave equation

In order to guarantee uniform velocity of propagation we need the exponent of the Fractional laplacian to be greater than 1.

Higher order Fractional laplacian

$$(-\Delta)^{s+1} := (-\Delta)^s(-\Delta)$$

$$\mathcal{D}((-\Delta)^{s+1}) = H^3(\Omega) \cap H_0^{2(s+1)}(\Omega)$$

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where ω is a neighbourhood of the boundary of the domain and χ is the characteristic function.

Theorem

Let $T > 0$ and $\omega \subset \Omega$ be a neighbourhood of the boundary of the domain. Then, for any couple of initial data $(y_0, y_1) \in H_0^1(\Omega)\times L^2(\Omega)$ there exists $h \in L^2(\omega\times[0,T])$ such that the solution of (5) satisfies $y(x,T) = y_t(x,T) = 0$.

Proposition

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it holds

$$\begin{aligned} \Gamma(1+s)^2 \int_\Sigma \left(\frac{-\Delta u}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt &= \frac{2s-n}{2} \int_Q \left((-\Delta)^{\frac{s+2}{2}} u \right)^2 dxdt \\ &+ \frac{2+n}{2} \int_Q (\nabla u_t)^2 \, dxdt \\ &+ \int_\Omega u_t (x \cdot \nabla(-\Delta u)) dx \Big|_0^T \end{aligned}$$

$$\Sigma := \partial\Omega \times [0, T] \qquad \delta = \delta(x) := d(x, \partial\Omega)$$

Proof.

The identity is obtained by multiplying the equation by $x \cdot \nabla(-\Delta u)$ and integrating over Q by using the Pohozaev identity for the fractional laplacian (4) applied to $(-\Delta u)$. □

Theorem (Energy estimate)

For any solution of (2) we define the energy as

$$E(t) := \frac{1}{2} \int_\Omega \left\{ (\nabla u_t)^2 + \left((-\Delta)^{\frac{s+2}{2}} u \right)^2 \right\} dx;$$

for any $T > 0$ there exists two non negative constants A_1 and A_2, depending only on s, T and Ω, such that

$$A_1 E_0 \leq \int_\Sigma \left(\frac{-\Delta u}{\delta^s} \right)^2 (x \cdot \nu) d\sigma dt \leq A_2 E_0$$

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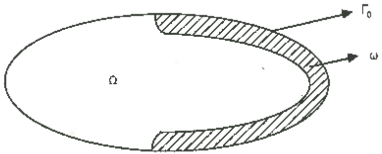

We will use Hilbert Uniqueness Method

Observability inequality

$$E_0 \leq C \int_0^T \int_\omega |u|^2 dxdt$$

Proof of the observability inequality

1. $E_0 \leq C_1 \int_0^T \int_{\hat{\omega}} \left\{ (\nabla u_t)^2 + \left((-\Delta)^{\frac{s+2}{2}} u \right)^2 \right\} dxdt \qquad (\Omega \cap \overline{\hat{\omega}}) \subset \omega$
2. equipartition of the energy $\Rightarrow E_0 \leq C_2 \int_0^T \left\| (-\Delta)^{\frac{s+2}{2}} u \right\|^2_{L^2(\omega)} dt$
3. $E_0 \leq C_2 \int_0^T \|u\|^2_{H^{s+2}(\omega)} dt$
4. The proof is concluded by interpolation.

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Control result via HUM

$$\begin{cases} u_{tt} + (-\Delta)^{s+1} u = 0 \\ u|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$$

BACKWARD SYSTEM

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The solution of the backward system is defined by transposition: the function y is a solution of the problem if and only if for any solution θ of

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$$\int_Q \psi f dxdt - \int_\Omega (\psi_t(0)\theta_0 - \psi(0)\theta_1) dx = \int_Q \theta h dxdt \qquad (6)$$

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by considering (6) with $\theta = u$ and choosing the control function

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$$\langle \Lambda(u_0, u_1); (u_0, u_1)\rangle = \int_0^T \int_\omega |u|^2 dxdt.$$

Observability inequality $\Rightarrow$
$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$ is an isomorphism.

For any couple of initial data $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a unique solution $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ of $\Lambda(u_0, u_1) = (y_1, -y_0)$

The control function $h \in L^2(0, T; L^2(\omega))$ drives the system in rest in time T.

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Work in progress

Pohozaev identity for the fractional laplacian $(-\Delta)^{s+1}u$

$$\begin{aligned}\int_\Omega (-\Delta)^{s+1}u\,(x\cdot\nabla u)\,dx &= \frac{2(s+1)-n}{2}\int_\Omega u(-\Delta)^{s+1}u dx \\ &- \frac{\Gamma(2+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^{s+1}}\right)^2(x\cdot\nu)d\sigma\end{aligned} \tag{7}$$

Starting from (7) it is possible to repeat the analysis just presented and prove the control result for the fractional wave equation.

The observability inequality is proved with no need of interpolation techniques.

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Fourier analysis

In order to guarantee an uniform velocity of propagation we need $s \geq 1/2$ in the Schrödinger equation and $s \geq 1$ in the wave equation.

1-d fractional problems on $(-1,1) \times (0,T)$

$$\begin{cases} iu_t + (-d_x^2)^{1/2}u = 0 \\ u|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ u(x,0) = u_0(x) \end{cases} \implies u(x,t) = \sum_{k \geq 1} a_k \phi_k(x) e^{i\mu_k t}$$

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$\{\mu_k, \phi_k(x)\}_{k \geq 1}$ are the eigenvalues and the eigenfunctions of the square root of the laplacian in 1-d on the interval $(-1,1)$.

Fourier analysis

In order to guarantee an uniform velocity of propagation we need $s \geq 1/2$ in the Schrödinger equation and $s \geq 1$ in the wave equation.

1-d fractional problems on $(-1,1) \times (0,T)$

$$\begin{cases} iu_t + (-d_x^2)^{1/2} u = 0 \\ u\big|_{\mathbb{R}^n \setminus \Omega} \equiv 0 \\ u(x,0) = u_0(x) \end{cases} \implies u(x,t) = \sum_{k \geq 1} a_k \phi_k(x) e^{i\mu_k t}$$

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Spectral analysis for the square root of the Laplacian

T. KULCZYCKI, M. KWAŜNICKI, J. MALECKI and E. STOS - Spectral properties of the Cauchy process on half-line and interval.

$$\left|\mu_k - \left(\frac{k\pi}{2} - \frac{\pi}{8}\right)\right| = O\left(\frac{1}{k}\right) \quad \textit{for} \quad k \geq 1$$

$$\left\|\phi_k - \textit{sin}\left(\left(\frac{k\pi}{2} - \frac{\pi}{8}\right)(1+x) + \frac{\pi}{8}\right)\right\|_{L^2(-1,1)} = O\left(\frac{1}{\sqrt{k}}\right) \quad \textit{for} \quad k \geq 1$$

Gap between the eigenvalues

Schrödinger equation

$$\mu_{k+1} - \mu_k \to \left[\frac{(k+1)\pi}{2} - \frac{\pi}{8}\right] - \left[\frac{k\pi}{2} - \frac{\pi}{8}\right] = \frac{\pi}{2}$$

Wave equation

$$\sqrt{\mu_{k+1}} - \sqrt{\mu_k} \to \sqrt{\frac{(k+1)\pi}{2} - \frac{\pi}{8}} - \sqrt{\frac{k\pi}{2} - \frac{\pi}{8}} \to 0$$

General case $\beta \in (0,1)$

$$(-d_x^2)^\beta \Rightarrow \mu_k = \left(\frac{k\pi}{2} - \frac{(2-2\beta)\pi}{8}\right)^{2\beta} + O\left(\frac{1}{k}\right)$$

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THANKS FOR YOUR ATTENTION!