Parabolic quasiminimizers

Jens Habermann

University of Erlangen

Partial differential equations, optimal design and numerics
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Plan of the talk:

- Introduction: Parabolic quasiminimizers
- Stability and Higher Integrability
- Extensions to metric measure spaces

Papers:
Introduction: Parabolic quasimimimizers
Parabolic quasiminimizers: the model case

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $\Omega_T := \Omega \times (0, T)$, $T > 0$, and let $u \in L^p(0, T; W^{1,p}(\Omega))$ satisfy

$$-\int_{\text{spt} \, \varphi} u \partial_t \varphi \, dz + \frac{1}{p} \int_{\text{spt} \, \varphi} |D u|^p \, dz \leq \frac{Q}{p} \int_{\text{spt} \, \varphi} |D u - D \varphi|^p \, dz,$$

for all $\varphi \in C_c^\infty(\Omega_T)$, $Q \geq 1$, under suitable initial and boundary conditions on $\partial \text{par} \, \Omega_T = (\Omega_T \times \{0\}) \cup (\partial \Omega \times (0, T))$. We denote: $\partial_t \varphi \equiv \partial \varphi / \partial t$ and $D \varphi \equiv \nabla x \varphi$. $\Omega \times \{0\}$ is called a parabolic $Q$-minimizer of the $p$-energy.
Parabolic quasimimimizers: the model case

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Parablic equations and parabolic quasiminimizers

- Weak solutions of the parabolic $p$-Laplace equation

  \[ u_t - \text{div} \left( |Du|^{p-2} Du \right) = 0, \quad \text{on } \Omega_T \]

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$$\partial_t u - \text{div} A(x, t, Du) = 0, \quad \text{on } \Omega_T$$

with polynomial growth

$$A(x, t, \xi) \cdot \xi \geq \lambda |\xi|^p, \quad |A(x, t, \xi)| \leq L |\xi|^{p-1},$$

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- Quasiminimizers $\leftrightarrow$ Obstacle problems

- Other examples by Wieser ('87, Manus. Math.), Zhou ('93/'94, J. PDE)
Stability and Higher Integrability
Problem: Take sequences \( p_i \to p \) and \( Q_i \to Q \) and consider a sequence \( u_i \) of parabolic \( Q_i \)-minimizers of the \( p_i \)-energy with fixed boundary data \( u_i = \eta \) on \( \partial_{\text{par}} \Omega_T \), for which holds

\[
u_i(x, t) \to u(x, t) \text{ a.e. on } \Omega_T.
\]

Is \( u \) a parabolic \( Q \)-minimizer of the \( p \)-energy with \( u = \eta \) on \( \partial_{\text{par}} \Omega_T \)?

Answer: In general NO! Counterexamples by Lindqvist, Kilpeläinen & Koskela, Kinnunen & Parviainen for the parabolic \( p \)-Laplace. Strongly related to the fact that in general \( W^{1, p}(\Omega) \cap W_{o\text{-loc}}^{1, p}(\Omega) \neq W_{o\text{-loc}}^{1, p}(\Omega) \).

Answer is YES for regular domains [Hedberg, Kilpeläinen, '99].
Stability of Parabolic quasiminimizers

**Problem:** Take sequences $p_i \to p$ and $Q_i \to Q$ and consider a sequence $u_i$ of parabolic $Q_i$-minimizers of the $p_i$-energy with fixed boundary data $u_i = \eta$ on $\partial_{\text{par}} \Omega_T$, for which holds

$$ u_i(x, t) \to u(x, t) \text{ a.e. on } \Omega_T. $$

Is $u$ a parabolic $Q$-minimizer of the $p$-energy with $u = \eta$ on $\partial_{\text{par}} \Omega_T$?

**Answer:** In general NO!

Counterexamples by Lindqvist, Kilpeläinen & Koskela, Kinnunen & Parviainen for the parabolic $p$-Laplace

Strongly related to the fact that in general

$$ W^{1,p}(\Omega) \cap W_0^{1,p-\varepsilon}(\Omega) \neq W_0^{1,p}(\Omega). $$
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The stability problem

Theorem (Fujishima, H., Kinnunen, Masson, Potential Anal., ’14):

Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick, $p \geq 2$ and $\eta \in C^1(\Omega_T)$ be fixed.

Let $\{p_i\}_i$ and $\{Q_i\}_i$ be two sequences with $p_i \to p \geq 2$ and $Q_i \to Q \geq 1$ as $i \to \infty$.

Consider a sequence $u_i \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$ of parabolic $Q_i$-minimizers of the $p_i$-energy with $u_i = \eta$ on $\partial_{\text{par}} \Omega_T$ and

$$u_i(x, t) \to u(x, t) \text{ almost everywhere in } \Omega_T.$$

Then

$$u \in L^p(0, T; W^{1,p}(\Omega)),$$

and $u$ is a parabolic $Q$-minimizer of the $p$-energy with $u = \eta$ on $\partial_{\text{par}} \Omega_T$. 
A very rough sketch of the proof

- Note: If $p$ changes, then the space $L^p(0, T; W^{1,p}(\Omega))$ changes!
  $\rightarrow$ Establish uniform energy bounds.
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**Lemma (H., Nonlin. Anal., 2014):** There exists a universal $\varepsilon > 0$, such that

$$\int_{\Omega_T} |Du|^{p+\varepsilon} \, dz < \infty.$$  

- Parab. $p$-Laplace, $p \geq 2$: Parviainen ('08)
- Parab. $Q$-min, $p = 2$: Masson & Parviainen ('14)
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**Remarks on the Proof:**

- Global result. It uses the regularity of the boundary.
- Self-improving property of uniform $p$-thickness [Lewis, '88]
- Intrinsic geometry [DiBenedetto, Friedman, '85]
Some remarks on the stability proof

- Establish convergence $u_i \to u$.
  - Higher integrability $\implies$ Uniform energy bound
    \[
    \sup_{i \in \mathbb{N}} \int_{\Omega_T} |u_i|^{p+\delta} + |Du_i|^{p+\delta} \, dz < \infty,
    \]
  - Compactness argument (J. Simon, '87) provides that the limit function
    \[
    u \in L^{p+\delta}(0, T; W^{1,p+\delta}(\Omega)),
    \]
    and for a subsequence we get $u_i \to u$ strongly in $L^{p+\delta}(\Omega_T)$ and $Du_i \to Du$ weakly in $L^{p+\delta}(\Omega_T)$.

- $u$ attains the initial and lateral boundary data $\eta$
  - Use uniform energy bounds, uniform $p$-thickness of $\mathbb{R}^n \setminus \Omega$.
  - Self-improving property of uniform $p$-thickness [Lewis, '88].
  - Use characterization of boundary values for Sobolev functions

- $u$ is a parabolic $Q$-minimizer of the $p$-energy
  - Delicate argument, using again the uniform energy bounds
Some remarks on the stability proof

- Simple, direct proof, using merely
  - Energy estimates for the functions $u_i$ and their gradients $Du_i$;
  - General properties and embeddings for Sobolev functions;
  - Characterizations of Sobolev functions at the boundary;
  - the self-improving property of $p$-thickness.

- In particular, the proof does not use
  - Uniqueness, comparison or maximum principles
  - A priori informations on the limit functions

Literature:
- Stationary case:
  - Kilpeläinen & Koskela ('94, Nonlin. Anal.)
  - Li & Martio ('98, Indiana Univ. Math. J.)
  - Zhikov ('97, Russian J. Math. Phys.)
- Time-dependent case:
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- Time-dependent case:
Extension to metric measure spaces
Parabolic quasiminimizers on metric measure spaces

**Now:** Replace $\mathbb{R}^n$ by a *metric measure space* $(\mathcal{X}, d, \mu)$.

How can we define a 'gradient' $\nabla u$ for a function $u : \mathcal{X} \to \mathbb{R}$?
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How can we define a 'gradient' $\nabla u$ for a function $u: \mathcal{X} \to \mathbb{R}$?

Recall: Characterization of Sobolev functions by means of path integrals: For $u \in W^{1,p}(\Omega)$ there holds

$$|u(x) - u(y)| \leq \int_{\gamma} |\nabla u| \, ds,$$

for $p$-almost all rectifiable curves $\gamma$ (parametrized by arclength) connecting $x$ and $y$. 
Parabolic quasiminimizers on metric measure spaces

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Recall: Characterization of Sobolev functions by means of path integrals: For $u \in W^{1,p}(\Omega)$ there holds

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for $p$-almost all rectifiable curves $\gamma$ (parametrized by arclength) connecting $x$ and $y$.

**Definition:** A Borel measurable function $g : \mathcal{X} \to [0, \infty]$ is called an upper gradient of $u : \mathcal{X} \to \mathbb{R}$, if

$$\left| u(\gamma(\ell_\gamma)) - u(\gamma(0)) \right| \leq \int_{\gamma} g \, ds,$$

for all rectifiable curves $\gamma : [0, \ell_\gamma] \to \mathcal{X}$.
Upper gradients in metric spaces

**Minimal upper gradient:** Defined by the property that

\[ \|g_u\|_{L^p(\Omega)} = \inf_G \|g\|_{L^p(\Omega)} \]

where \( G \) denotes the set of all upper gradients \( g \in L^p(\Omega) \).

\[ \to \quad \text{Analog concept to the one of Sobolev spaces: Newtonian space} \]

\[ \mathcal{N}^{1,p}(\Omega). \]

[Cheeger, Hajlasz, Shanmugalingam]
Parabolic quasi minimizers on \((\mathcal{X}, d, \mu)\)

Given a metric measure space \((\mathcal{X}, d, \mu)\), \(\Omega \subset \mathcal{X}\) open and \(\Omega_T \equiv \Omega \times (0, T) \subset \mathcal{X} \times \mathbb{R}\).

Consider \(u: \Omega_T \rightarrow \mathbb{R}^N, N \geq 1\), satisfying

\[
\iint_{\text{spt} \varphi} u \partial_t \varphi \, d\mu \, dt + \iint_{\text{spt} \varphi} g_u^p \, d\mu \, dt \leq Q \iint_{\text{spt} \varphi} g_{u-\varphi}^p \, d\mu \, dt,
\]

for all testing functions \(\varphi \in \text{Lip}_c(\Omega_T)\), with \(Q \geq 1\) fixed.
Parabolic quasi minimizers on \((X, d, \mu)\)

Given a metric measure space \((X, d, \mu)\), \(\Omega \subset X\) open and \(\Omega_T \equiv \Omega \times (0, T) \subset X \times \mathbb{R}\).

Consider \(u : \Omega_T \to \mathbb{R}^N, N \geq 1\), satisfying

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for all testing functions \(\varphi \in \text{Lip}_c(\Omega_T)\), with \(Q \geq 1\) fixed.

Assumptions on the metric measure space:

- \((X, d, \mu)\) is doubling, i.e.

\[
\frac{\mu(B_{2R}(x))}{\mu(B_R(x))} \leq C, \quad \text{for all } B_{2R}(x) \subset X.
\]

- \((X, d, \mu)\) supports a \((1, p)\)-Poincaré inequality

\[
\int_{B_{\varrho}(x_0)} |u - u_{x_0, \varrho}| \, d\mu \leq c_p \varrho \left[ \int_{B_{2\varrho}(x_0)} g^p \, d\mu \right]^{1/p}
\]
Motivation and Examples

Examples:

- Some weighted Euclidean spaces [Heinonen, Kilpeläinen, Martio, 1993]
- Classes of Riemannian manifolds [Saloff-Coste, 2002]
- Some Ahlfors $Q$-regular spaces [Laakso, 2000]
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Goals:

- Regularity for upper gradients
- Poincaré inequality $\iff$ Harnack inequalities
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Goals:

- Regularity for upper gradients
- Poincaré inequality $\iff$ Harnack inequalities

Obstacles:

- No PDEs, fundamental solution, comparison arguments, ...
- Upper gradients are not linear
- No standard approximation procedures
Towards stability in metric measure spaces

Global higher integrability of Gehring type:

Theorem (Fujishima, H., Preprint):
Let $\Omega \subset \mathcal{X}$ be a 'regular' domain, $u \in L^p(0, T; \mathcal{N}^{1,p}(\Omega))$ be a parabolic quasi minimizer with boundary values $u = \eta$ on $\partial_{\text{par}} \Omega_T$. Then there exists $\varepsilon > 0$, depending only on the structure parameters of the problem, such that

$$u \in L^{p+\varepsilon}(0, T; \mathcal{N}^{1,p+\varepsilon}(\Omega)).$$
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- Euclidean setting ($X = \mathbb{R}^n$):
  - Wieser '87, Kinnunen & Lewis '00, Bögelein '08, Bögelein & Parviainen '10, Bögelein & Duzaar '11, H. '14

- Metric spaces ($X, d, \mu$):
  - Masson, Miranda, Paronetto, Parviainen '13 ($p = 2$, local)
  - Masson, Parviainen '15 ($p = 2$, up-to-the-boundary)
  - H. '14 ($p \neq 2$, local)
Literature on quasiminimizers on metric measure spaces:

- **Metric measure spaces, properties:** Cheeger, Saloff-Coste, Lewis, Keith, Zhong, Koskela,…

- **Elliptic problems studied in the past 10-15 years:** Kinnunen, Shanmugalingam, Björn, Marola, Koskela, MacManus, Maasalo, Lindqvist, Zatorska-Goldstein,…

- **Parabolic problems on metric measure spaces:** Saloff-Coste, Grigoryan, Kinnunen, Kilpeläinen, Koskela, Marola, Miranda, Paronetto, Masson, Parviainen, Siljander,…

Many techniques also come from the study of parabolic problems in the Euclidean setting: DeGiorgi, Nash, Giusti, DiBenedetto, Gianazza, Vespri, Wieser, Duzaar, Bögelein, Zhou, …

Thank you for your attention.