Brocket-Wegner Flow and Diagonalization of Quadratic Operators in Boson Quantum Field Theory

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Joint work with V. Bach
To appear in Memoirs of the AMS, 2014

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Outline

1. Quadratic Boson Operators
2. Brocket–Wegner Flow Equations
4. Concluding Remarks
One–Particle Hilbert Space

- $\mathcal{H} := L^2(\mathcal{M}, \alpha)$ is a separable complex Hilbert space of square–integrable functions on a measure space $(\mathcal{M}, \alpha)$. The scalar product on $\mathcal{H}$ is given by

$$\langle f | g \rangle := \int_{\mathcal{M}} \overline{f(x)} g(x) \, d\alpha(x).$$

- For any operator $X$ on $\mathcal{H}$, we define its transpose $X^t$ and its complex conjugate $\overline{X}$ by $\langle f | X^t g \rangle := \langle \overline{g} | X \overline{f} \rangle$ and $\langle f | \overline{X} g \rangle := \overline{\langle \overline{f} | X \overline{g} \rangle}$, for $f, g \in \mathcal{H}$, respectively. Note that $X^* = \overline{X}^t = \overline{X}^t$.

- $\mathcal{B}(\mathcal{H})$ is the Banach space of bounded operators acting on $\mathcal{H}$ and $L^2(\mathcal{H})$ is the Hilbert spaces of Hilbert–Schmidt operators defined from the scalar product

$$(X, Y)_2 := \text{trace}(X^* Y) \quad \text{with} \quad \|X\|_2 := \text{trace}(X^* X).$$

- Condition A1: Let $\Omega_0 = \Omega_0^* \geq 0$ be a positive (possibly unbounded) operator on $\mathcal{H}$.

- Condition A2: Let $B_0 = B_0^t \in L^2(\mathcal{H})$ be a (non–zero) Hilbert–Schmidt operator.
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Many–particle Hilbert Space: The Boson Fock Space

- The boson Fock space is the Hilbert space
  \[ \mathcal{F}_b := \bigoplus_{n=0}^{\infty} S_n (\mathfrak{h}^\otimes n). \]
  Here, \( S_n \) is the orthogonal projection onto the subspace of totally symmetric \( n \)-particle wave functions in \( \mathfrak{h}^\otimes n \), the \( n \)-fold tensor product of \( \mathfrak{h} \).

- Annihilation/Creation operators \( \{a(f), a^*(f)\} \) are unbounded operators on \( \mathcal{F}_b \) defined for \( f \in \mathfrak{h} \) by
  \[
  (a(f) \psi)^{(n)}(x_1, \ldots, x_n) := \sqrt{n+1} \langle f | (\psi)^{(n+1)}(\cdot, x_1, \ldots, x_n) \rangle \\
  (a^*(f) \psi)^{(n)}(x_1, \ldots, x_n) := S_n \left( f(x) (\psi)^{(n-1)}(x_1, \ldots, x_{n-1}) \right)
  \]
  They satisfy the Canonical Commutation Relation (CCR):
  \[
  [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \quad \text{whereas} \quad [a(f), a^*(g)] = \langle f | g \rangle.
  \]
  Here
  \[
  [A, B] := AB - BA.
  \]
Quadratic Boson Operators

- **Condition A1**: Let $\Omega_0 = \Omega_0^* \geq 0$ with domain $\mathcal{D}(\Omega_0) \subseteq \mathfrak{h}$.
- **Condition A2**: Let $B_0 = B_0^t \in \mathcal{L}^2(\mathfrak{h})$ be a (non-zero) Hilbert–Schmidt operator.

Take some orthonormal basis $\{\varphi_k\}_{k=1}^{\infty} \subset \mathcal{D}(\Omega_0) \subseteq \mathfrak{h}$ and let $a_k := a(\varphi_k)$.

Then, for any fixed $C_0 \in \mathbb{R}$, the quadratic boson operator is defined by

$$H_0 := \sum_{k,\ell} \{\Omega_0\}_{k,\ell} a_k^* a_\ell + \{B_0\}_{k,\ell} a_k^* a_\ell^* + \{\bar{B}_0\}_{k,\ell} a_k a_\ell + C_0$$

with $\{X\}_{k,\ell} := \langle \varphi_k | X \varphi_\ell \rangle$. ($\int dk \, d\ell$ could also replace $\sum_{k,\ell}$.)

**Proposition (Berezin (66) – Bruneau-Derezinski (07))**

*Under Conditions A1–A2, $H_0$ is essentially self–adjoint on the domain*

$$\mathcal{D}(H_0) := \bigcup_{N=1}^{\infty} \left( \bigoplus_{n=0}^{N} S_n \left( \mathcal{D}(\Omega_0)^\otimes n \right) \right) \subseteq \mathcal{F}_b.$$
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A2 Let \( B_0 = B_0^t \in \mathcal{L}^2(\mathfrak{h}) \) be a (non–zero) Hilbert–Schmidt operator.
A3 \( \Omega_0 \geq 4B_0(\Omega_0^t)^{-1}\bar{B}_0 \).
B4 \( \Omega_0^{-1-\varepsilon}B_0 \in \mathcal{L}^2(\mathfrak{h}) \) and \( 1 \geq (4 + r) B_0(\Omega_0^t)^{-2}\bar{B}_0 \) for some constant \( r, \varepsilon > 0 \).
B5 \( \Omega_0 \geq 4B_0(\Omega_0^t)^{-1}\bar{B}_0 + \mu 1 \) for some constant \( \mu > 0 \).

**Theorem (Bach-B (14))**

Under A1–A3 and either B4 or B5, there are \( \Omega_\infty = \Omega_\infty^* \geq 0 \) on \( \mathfrak{h} \), \( C_\infty \in \mathbb{R} \) and a unitary operator \( U_\infty \) on \( \mathcal{F}_b \) such that

\[ U_\infty H_0 U_\infty^* = H_\infty := \sum_{k,\ell} \{\Omega_\infty\}_{k,\ell} a_k^* a_\ell + C_\infty \]

Then \( H_\infty \) can be diagonalized by a unitary operator acting on \( \mathfrak{h} \), only.
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Historical Overview on Diagonalization of Such Operators

- 1947, Bogoliubov: $\Omega_0$ and $B_0$ are $2 \times 2$ real matrices satisfying A1–A2, B5 and

  $\Omega_0 B_0 = B_0 \Omega_0$.

  $\Omega_\infty$ and $C_\infty$ are explicitly known. Assumptions stronger than A1, A2, and B5.
  
- 1953, Friedrichs - 1966, Berezin: $\Omega_0 \in B(\mathfrak{h})$ and $B_0 \in L^2(\mathfrak{h})$ are both real
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  \text{trace} \left( \Omega_\infty^2 - \Omega_0^2 + 4B_0 \bar{B}_0 \right) = 0 \quad \text{and} \quad C_\infty = C_0 + \frac{1}{2} \text{trace} \left( \Omega_0 - \Omega_\infty \right).
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  Furthermore, if $\Omega_0 B_0 = B_0 \Omega_0^t$ then

  $\Omega_\infty = \left\{ \Omega_0^2 - 4B_0 \bar{B}_0 \right\}^{1/2}$ \quad \text{and} \quad $C_\infty = C_0 + \frac{1}{2} \text{trace} \left( \Omega_0 - \left\{ \Omega_0^2 - 4B_0 \bar{B}_0 \right\}^{1/2} \right)$.
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Strategy of the Proof

- We use a \textit{(quadratically) nonlinear first–order} differential equation:

\[ \forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0, \quad A = N := \sum_k a_k^* a_k. \]
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- Then explicit computations using the CCR to study the commutators show formally that

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H_t := \sum_{k, \ell} \{\Omega_t\}_{k, \ell} a^*_k a_\ell + \{B_t\}_{k, \ell} a^*_k a^*_\ell + \{\bar{B}_t\}_{k, \ell} a_k a_\ell + C_0 + 8 \int_0^t \|B_\tau\|^2 d\tau,
\]

where the operators \(\Omega_t = \Omega_t^*\) and \(B_t = B_t^t\) satisfy a system of (quadratically) nonlinear first–order differential equations

\[
\forall t \geq 0 : \quad \begin{cases}
\partial_t \Omega_t = -16B_t \bar{B}_t, \\
\partial_t B_t = -2 \left(\Omega_t B_t + B_t \Omega_t^t\right),
\end{cases} \quad \Omega_{t=0} := \Omega_0, \quad B_{t=0} := B_0,
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  \end{cases}
  \]

- **The map** \(t \mapsto \|B_t\|_2\) **must be, at least, square–integrable on** \([0, \infty)\). **Then in the strong resolvent sense,**
  \[
  H_\infty := \sum_{k, \ell} \{\Omega_\infty\}_{k, \ell} a_k^* a_\ell + C_\infty = \lim_{t \to \infty} H_t
  \]
Let the unitary operator $U_{t,s}$ on a Hilbert space $\mathcal{H}$ be the solution of the non-autonomous evolution equation

$$\forall t \geq s \geq 0 : \quad \partial_t U_{t,s} = -iG_t U_{t,s}, \quad U_{s,s} := 1,$$

with self-adjoint (s.a.) generator $G_t$. Here $U_t := U_{t,0}$.

**Remark:** There is no unified theory of such Cauchy problem for unbounded generators $G_t$ in spite of its long history starting 60 years ago.
Flow Equations for Operators

1. Let the unitary operator \( U_{t,s} \) on a Hilbert space \( \mathcal{H} \) be the solution of the non-autonomous evolution equation

\[
\forall t \geq s \geq 0 : \quad \partial_t U_{t,s} = -iG_t U_{t,s}, \quad U_{s,s} := 1,
\]

with self-adjoint (s.a.) generator \( G_t \). Here \( U_t := U_{t,0} \).

**Remark:** There is no unified theory of such Cauchy problem for unbounded generators \( G_t \) in spite of its long history starting 60 years ago.

2. Let \( H_0 = H_0^* \) acting on \( \mathcal{H} \). Then \( H_t := U_t H_0 U_t^* \) satisfies

\[
\forall t \geq 0 : \quad \partial_t H_t = i [H_t, G_t] := i(H_t G_t - G_t H_t), \quad H_{t=0} := H_0.
\]
Flow Equations for Operators

1. Let the unitary operator $U_{t,s}$ on a Hilbert space $\mathcal{H}$ be the solution of the non-autonomous evolution equation

$$\forall t \geq s \geq 0 : \quad \partial_t U_{t,s} = -iG_t U_{t,s}, \quad U_{s,s} := 1,$$

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2. Let $H_0 = H_0^*$ acting on $\mathcal{H}$. Then $H_t := U_t H_0 U_t^*$ satisfies

$$\forall t \geq 0 : \quad \partial_t H_t = i[H_t, G_t] := i(H_t G_t - G_t H_t), \quad H_{t=0} := H_0.$$

**Question:** find $G_t$ depending on a fixed operator $A$ such that in the limit $t \to \infty$,

$$H_\infty = U_\infty H_0 U_\infty^*, \quad \text{with} \quad [A, H_\infty] := AH_\infty - H_\infty A = 0.$$
Assume that $H_0 = H_0^*$ and $A = A^*$ are two self-adjoint matrices.

Let

$$\forall t \geq 0 : \quad f(t) := \text{trace}((H_t - A)^2) = \|H_t - A\|_2^2 \geq 0$$

and observe that

$$\partial_t f(t) = \partial_t \left\{ \text{trace} \left( H_t^2 - 2H_tA + A^2 \right) \right\} = \partial_t \{ \text{trace} (-2H_tA) \} = -2\text{trace} (i[H_t, G_t]A) = -2\text{trace} (i[A, H_t] G_t),$$

by using $\partial_t H_t = i[H_t, G_t]$ and the cocyclicity of the trace.
Assume that $H_0 = H_0^*$ and $A = A^*$ are two self-adjoint matrices.

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$$\forall t \geq 0 : \quad f(t) := \text{trace}((H_t - A)^2) = \|H_t - A\|_2^2 \geq 0$$

and observe that

$$\partial_t f(t) = \partial_t \left\{ \text{trace} \left( H_t^2 - 2H_tA + A^2 \right) \right\} = \partial_t \left\{ \text{trace} \left( -2H_tA \right) \right\}$$

$$= -2\text{trace} \left( i[H_t, G_t]A \right) = -2\text{trace} \left( i[A, H_t]G_t \right),$$

by using $\partial_t H_t = i[H_t, G_t]$ and the cocyclicity of the trace.

**Choice of the generator:**

$$G_t := i[A, H_t] := i(AH_t - H_tA).$$
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\[
\forall t \geq 0 : \quad f(t) := \text{trace}((H_t - A)^2) = \|H_t - A\|_2^2 \geq 0
\]
and observe that
\[
\partial_t f(t) = \partial_t \{\text{trace} (H_t^2 - 2H_tA + A^2)\} = \partial_t \{\text{trace} (-2H_tA)\}
\]
\[
= -2\text{trace} (i [H_t, G_t] A) = -2\text{trace} (i [A, H_t] G_t),
\]
by using \( \partial_t H_t = i [H_t, G_t] \) and the cocyclicity of the trace.

**Choice of the generator:** \( G_t := i [A, H_t] := i (AH_t - H_tA) \).

We then obtain
\[
\forall t \geq 0 : \quad \partial_t f(t) = -2\text{trace} (G_t G_t^*) = -2\|G_t\|_2^2 \leq 0 .
\]
Assume that $H_0 = H_0^*$ and $A = A^*$ are two self-adjoint matrices.

Let

$$\forall t \geq 0 : \quad f(t) := \text{trace}((H_t - A)^2) = \|H_t - A\|_2^2 \geq 0$$

and observe that

$$\partial_t f(t) = \partial_t \left\{ \text{trace} \left( H_t^2 - 2H_tA + A^2 \right) \right\} = \partial_t \left\{ \text{trace}(-2H_tA) \right\} = -2\text{trace}(i[H_t, G_t]A) = -2\text{trace}(i [A, H_t] G_t),$$

by using $\partial_t H_t = i [H_t, G_t]$ and the cocyclicity of the trace.

**Choice of the generator:**

$$G_t := i [A, H_t] := i (AH_t - H_tA).$$

1. We then obtain

$$\forall t \geq 0 : \quad \partial_t f(t) = -2\text{trace}(G_t G^*_t) = -2\|G_t\|_2^2 \leq 0 .$$

2. This suggests that $\partial_t f(t) \to 0$ as $t \to \infty$, which implies $i[A, H_t] \to 0$ and that

$$H_t = U_t H_0 U^*_t \to H_\infty = U_\infty H_0 U^*_\infty \quad \text{with} \quad [A, H_\infty] = 0 .$$
The Brocket–Wegner flow has successfully been applied to various problems in Condensed Matter Physics including:

- Electron-phonon coupling
- Dissipative quantum systems
- Interacting fermions, Hubbard model
- Impurity problems
- Non-equilibrium systems

A similar idea is also used in quantum chromodynamics and quantum electrodynamics.
Brockett–Wegner Flow Equations

It is the \textit{(quadratically) nonlinear first–order} differential equation:

\[
\forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]] , \quad H_{t=0} := H_0.
\]


Mathematical difficulties of this idea:

1. Proof of the existence of \((H_t)_{t \geq 0}\) solution of the flow?

2. Problem of the existence of \((U_t)_{t \geq 0}\) such that \(H_t = U_t H_0 U_t^{-1}\) which means that

\[
\forall t \geq 0 : \quad \partial_t U_t = [A, H_t] U_t := -iG_t U_t , \quad U_t := 1.
\]

3. Proof of the existence of \(H_\infty = \lim_{t \to \infty} H_t\) ?

4. Proof of the existence of \(U_\infty = \lim_{t \to \infty} U_t\) ?

5. Proof of \(H_\infty = U_\infty H_0 U_\infty^*\) with \([H_\infty, A] = 0\) ?
Brocket–Wegner Flow for Operators

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$$\forall t \geq 0: \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0.$$ 


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### Brocket–Wegner Flow for Operators

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   \]

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Well-posedness of Brocket–Wegner Flow Equations

**Theorem (Bach-B ('10 - '14))**

Let \( \mathcal{H} \) be any separable Hilbert space.

- **Bounded operators: Global existence.** Take \( H_0 = H_0^* \), \( A = A^* \in \mathcal{B}(\mathcal{H}) \). Then there are two unique solutions \( (H_t)_{t \geq 0}, (U_t)_{t \geq 0} \in C^\infty([\mathbb{R}^+; \mathcal{B}(\mathcal{H})]) \) respectively of

\[
\forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0; \quad \partial_t U_t = [A, H_t] U_t, \quad U_0 := 1,
\]

and satisfying

\[
H_t = U_t H_0 U_t^*, \quad U_t^* U_t = U_t U_t^* = 1.
\]

- **Unbounded operators: Local existence.** The flow has a unique, smooth local unbounded solution \( \left(H_t = U_t H_0 U_t^* \right)_{t \in [0, T_{\text{max}})} \) under some restricted conditions on iterated commutators.

- **Blows up of the Brocket–Wegner flow.** There are two unbounded self–adjoint \( H_0 = H_0^*, A \geq 0 \) such that the flow has a (unbounded) local solution \( \left(H_t = U_t H_0 U_t^* \right)_{t \in [0, T_{\text{max}})} \) which blows up on its domain at a finite time \( T_{\text{max}} < \infty \).
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$$\forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0 ; \quad \partial_t U_t = [A, H_t] U_t, \quad U_0 := 1,$$

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  \forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0 ; \quad \partial_t U_t = [A, H_t] U_t, \quad U_0 := 1,
  \]
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  \[
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- **Blows up of the Brocket–Wegner flow.** There are two unbounded self–adjoint \( H_0 = H_0^*, A \geq 0 \) such that the flow has a (unbounded) local solution \((H_t = U_t H_0 U_t^*)_{t \in [0, T_{\text{max}})}\) which blows up on its domain at a finite time \( T_{\text{max}} < \infty \).
We use the Brocket–Wegner flow for any quadratic operators $H_0$:

$$\forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0, \quad A = N := \sum_k a_k^* a_k.$$ 

Then explicit computations using the CCR show formally that

$$H_t := \sum_{k, \ell} \{\Omega_t\}_{k,\ell} a_k^* a_\ell + \{B_t\}_{k,\ell} a_k^* a_\ell^* + \{\bar{B}_t\}_{k,\ell} a_k a_\ell + C_0 + 8 \int_0^t \|B_\tau\|_2^2 d\tau,$$

where the operators $\Omega_t = \Omega_t^*$ and $B_t = B_t^t$ must satisfy a system of (quadratically) nonlinear first–order differential equations

$$\forall t \geq 0 : \begin{cases} 
\partial_t \Omega_t = -16 B_t \bar{B}_t, \\
\partial_t B_t = -2 (\Omega_t B_t + B_t \Omega_t^t), \\
\Omega_{t=0} := \Omega_0, \\
B_{t=0} := B_0,
\end{cases}$$
We use the Brocket–Wegner flow for any quadratic operators $H_0$:

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where the operators $\Omega_t = \Omega_t^*$ and $B_t = B_t^t$ must satisfy a system of (quadratically) nonlinear first–order differential equations

$$\forall t \geq 0 : \quad \begin{cases} \partial_t \Omega_t = -16 B_t \bar{B}_t, & \Omega_{t=0} := \Omega_0, \\ \partial_t B_t = -2 (\Omega_t B_t + B_t \Omega_t^t), & B_{t=0} := B_0, \end{cases}$$

A blows up of the Brocket–Wegner flow can then be seen by taking

$$\Omega_0 = 0 \quad \text{and} \quad B_0 = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \quad \text{with} \quad b > 0.$$
Under Conditions A1–A3, there is \((\Omega_t, B_t)_{t \in [0, T_{\text{max}}]}\) solution of

\[
\forall t \in [0, T_{\text{max}}] : \begin{cases}
\partial_t \Omega_t = -16 B_t \bar{B}_t , \\
\partial_t B_t = -2 \left( \Omega_t B_t + B_t \Omega_t^t \right) ,
\end{cases} \quad \Omega_{t=0} := \Omega_0 , \quad B_{t=0} := B_0 .
\]

The operators \(\Omega_t = \Omega_t^* \geq 0\) and \(B_t \in L^2(\mathfrak{h})\) satisfies:

\[
\forall t \in [0, T_{\text{max}}) : \quad \text{trace} \left( \Omega_t^2 - 4 B_t \bar{B}_t - \Omega_0^2 + 4 B_0 \bar{B}_0 \right) = 0
\]

and if \(\Omega_0 B_0 = B_0 \Omega_0^t\) then

\[
\forall t \in [0, T_{\text{max}}) : \quad \Omega_t = \left\{ \Omega_0^2 - 4 B_0 \bar{B}_0 + 4 B_t \bar{B}_t \right\}^{1/2} .
\]

This is possible to show because

\[
\forall t \in [0, T_{\text{max}}) : \quad B_t = B_t^t = W_t B_0 W_t^t ,
\]

where \(W_t\) is a (bounded, positive) operator acting on \(\mathfrak{h}\) solution of

\[
\forall t \in [0, T_{\text{max}}) : \quad \partial_t W_t = -2 \Omega_t W_t , \quad W_t := 1 .
\]

So, one only has one equation solved by the contraction mapping principle.
Diagonalization of Quadratic Operators

1. Under Conditions A1–A3, there is \((\Omega_t, B_t)_{t \in [0, T_{\text{max}}]}\) solution of

\[
\forall t \in [0, T_{\text{max}}) \subseteq \mathbb{R}_0^+ : \quad \begin{cases} 
\partial_t \Omega_t = -16 B_t \bar{B}_t, & \Omega_{t=0} = \Omega_0, \\
\partial_t B_t = -2 (\Omega_t B_t + B_t \Omega_t^t), & B_{t=0} = B_0.
\end{cases}
\]

The operators \(\Omega_t = \Omega_t^* \geq 0\) and \(B_t \in L^2(\mathfrak{h})\) satisfies:

\[
\forall t \in [0, T_{\text{max}}) : \quad \text{trace} \left( \Omega_t^2 - 4 B_t \bar{B}_t - \Omega_0^2 + 4 B_0 \bar{B}_0 \right) = 0
\]

and if \(\Omega_0 B_0 = B_0 \Omega_0^t\) then

\[
\forall t \in [0, T_{\text{max}}) : \quad \Omega_t = \left\{ \Omega_0^2 - 4 B_0 \bar{B}_0 + 4 B_t \bar{B}_t \right\}^{1/2}.
\]

2. Existence of a unitary operator \(U_t\) as strong solution on \(D(N) \subsetneq F_b\) of

\[
\forall t \in [0, T_{\text{max}}) : \quad \partial_t U_t = [N, H_t] U_t := -i G_t U_t, \quad U_t := 1.
\]

Indeed, denoting \(\mathcal{Y} := B(D(N))\), the generator \(G_t := i [N, H_t] = G_t^*\) satisfies

\[
\|G_t\|_\mathcal{Y} \leq 11 \|B_0\|_2, \quad \|G_t - G_s\|_\mathcal{Y} \leq 11 \|B_t - B_s\|_2, \quad \|[N, G_t]\|_\mathcal{Y} \leq 22 \|B_0\|_2
\]
1 Under Conditions A1–A3, there is \((\Omega_t, B_t)_{t \in [0, T_{\max})}\) solution of

\[
\forall t \in [0, T_{\max}) \subset \mathbb{R}_0^+ : \quad \begin{cases} 
\partial_t \Omega_t = -16 B_t \bar{B}_t , & \Omega_{t=0} := \Omega_0 , \\
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\end{cases}
\]

The operators \(\Omega_t = \Omega^*_t \geq 0\) and \(B_t \in \mathcal{L}^2(\mathfrak{h})\) satisfies:

\[
\forall t \in [0, T_{\max}) : \quad \text{trace} \left( \Omega_t^2 - 4B_t \bar{B}_t - \Omega_0^2 + 4B_0 \bar{B}_0 \right) = 0
\]

and if \(\Omega_0 B_0 = B_0 \Omega_0^t\) then

\[
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\]

2 Existence of a unitary operator \(U_t\) as strong solution on \(\mathcal{D}(N) \subset \mathcal{F}_b\) of

\[
\forall t \in [0, T_{\max}) : \quad \partial_t U_t = [N, H_t] U_t := -i G_t U_t , \quad U_t := 1.
\]

3 Using resolvents, show next that, for all \(t \in [0, T_{\max})\),

\[
H_t := \sum_{k, \ell} \{ \Omega_t \}_{k, \ell} a^*_k a_{\ell} + \{ B_t \}_{k, \ell} a^*_k a^*_\ell + \{ \bar{B}_t \}_{k, \ell} a_k a_{\ell} + C_t = U_t H_0 U^*_t .
\]
Using an a priori estimate, the assumption \( \Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h}) \) exclude blows up, i.e., \( T_{\text{max}} = \infty \), and yields the convergence, in the strong resolvent sense, of the family \((H_t = U_t H_0 U_t^*)_{t \geq 0}\) of unbounded operators to

\[
H_\infty := \sum_{k, \ell} \{\Omega_\infty\} k, \ell \: \mathfrak{a}_k^* \mathfrak{a}_\ell + C_0 + 8 \int_0^\infty \|B_\tau\|_2^2 d\tau.
\]

In particular, one shows that

\[
\text{trace} \left( \Omega_t^2 - \Omega_0^2 + 4B_0 \overline{B}_0 \right) = 4\|B_t\|_2^2 \implies \text{trace} \left( \Omega_\infty^2 - \Omega_0^2 + 4B_0 \overline{B}_0 \right) = 0
\]

and if \( \Omega_0 B_0 = B_0 \Omega_0^t \) then

\[
\Omega_t = \left\{\Omega_0^2 - 4B_0 \overline{B}_0 + 4B_t \overline{B}_t \right\}^{1/2} \implies \Omega_\infty = \left\{\Omega_0^2 - 4B_0 \overline{B}_0 \right\}^{1/2}.
\]

Indeed, \( \Omega_0^{-1/2} B_0 \in \mathcal{L}^2(\mathfrak{h}) \) yields the integrability of \( t \mapsto \|B_t\|_2^2 \) on \([0, \infty)\) whereas

\[
\Omega_t = \Omega_0 - 16 \int_0^t B_\tau \overline{B}_\tau d\tau \implies \Omega_\infty = \Omega_0 - 16 \int_0^\infty B_\tau \overline{B}_\tau d\tau
\]

and

\[
\left\| \left\{ (H_\infty + i\lambda \mathbf{1})^{-1} - (H_t + i\lambda \mathbf{1})^{-1} \right\} (N + \mathbf{1})^{-1} \right\|_{\text{op}} \leq C \left( \int_0^\infty \|B_\tau\|_2^2 d\tau + \|B_t\|_2 \right).
\]
Using an a priori estimate, the assumption $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$ exclude blows up, i.e., $T_{\text{max}} = \infty$, and yields the convergence, in the strong resolvent sense, of the family $(H_t = U_t H_0 U_t^*)_{t \geq 0}$ of unbounded operators to

$$H_\infty := \sum_{k, \ell} \{\Omega_\infty\}_{k, \ell} a_k^* a_\ell + C_0 + 8 \int_0^\infty \|B_\tau\|_2^2 d\tau .$$

In particular, one shows that

$$\text{trace} \left( \Omega^2_t - \Omega^2_0 + 4B_0 \bar{B}_0 \right) = 4\|B_t\|_2^2 \implies \text{trace} \left( \Omega^2_\infty - \Omega^2_0 + 4B_0 \bar{B}_0 \right) = 0$$

and if $\Omega_0 B_0 = B_0 \Omega_0^t$ then

$$\Omega_t = \left\{ \Omega^2_0 - 4B_0 \bar{B}_0 + 4B_t \bar{B}_t \right\}^{1/2} \implies \Omega_\infty = \left\{ \Omega^2_0 - 4B_0 \bar{B}_0 \right\}^{1/2} .$$

Use Conditions B4 or B5 to obtain the strong convergence of $(U_t)_{t \geq 0}$ to $U_\infty$ as well as

$$H_\infty = U_\infty H_0 U_\infty^* .$$

Indeed, B4 (or B5), which yields $\Omega_0^{-1/2}B_0 \in \mathcal{L}^2(\mathfrak{h})$, implies the integrability of $t \mapsto \|B_t\|_2$ on $[0, \infty)$ whereas

$$\|(U_\infty - U_t)(N + 1)^{-1}\|_{\text{op}} \leq \tilde{C} \int_t^\infty \|B_\tau\|_2 d\tau . \quad (1)$$
Using an a priori estimate, the assumption $\Omega_0^{-1/2} B_0 \in L^2(\mathfrak{h})$ exclude blows up, i.e., $T_{\text{max}} = \infty$, and yields the convergence, in the strong resolvent sense, of the family $(H_t = U_t H_0 U_t^*)_{t \geq 0}$ of unbounded operators to

$$H_{\infty} := \sum_{k, \ell} \{\Omega_{\infty}\}_{k, \ell} a^*_k a_\ell + C_0 + 8 \int_0^\infty \|B_\tau\|^2_2 d\tau.$$  

In particular, one shows that

$$\text{trace} \left( \Omega_t^2 - \Omega_0^2 + 4B_0 \bar{B}_0 \right) = 4\|B_t\|^2_2 \implies \text{trace} \left( \Omega_{\infty}^2 - \Omega_0^2 + 4B_0 \bar{B}_0 \right) = 0$$

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Then $H_{\infty}$ can be diagonalized by a unitary operator acting on $\mathfrak{h}$, using a diagonalization of $\Omega_{\infty}$ to have

$$\{\Omega_{\infty}\}_{k, \ell} := \langle \varphi_k | \Omega_{\infty} \varphi_\ell \rangle = \delta_{k, \ell} \lambda_k.$$
The Brocket–Wegner flow \( \partial_t H_t = [H_t, [H_t, A]] \) can be a powerful technique as seen on quadratic operators. Easy to describe, but mathematically involved due to unboundedness of operators. See, e.g., [Bach-B ('14)] which has 122 pages.

3 sources of unboundedness in the quadratic operators \( H_0 \):

i. Unboundedness of creation/annihilation operators. Easily controlled. Not a problem either for all previous works on diagonalization of \( H_0 \).

ii. Unboundedness of \( \Omega_0 / \in \mathcal{B}(h) \). Controlled with the flow. Out of the scope of previous works.

iii. Unboundedness of \( \Omega_0^{-1} / \in \mathcal{B}(h) \). Controlled but absolutely nontrivial. Out of the scope of previous works.

This example is the first mathematical use of the Brocket–Wegner flow to diagonalize (unbounded) operators.
Concluding Remarks

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