from delay differential equations
to ordinary differential equations
(through partial differential equations)
dynamical systems and applications @ BCAM - Bilbao

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Outline

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- Dynamical systems [9,10]
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basics [9,12]

dynamical systems [9,10]

three different views

numerical analysis [1,5,11,15]
Given $d \geq 1$ and $\tau > 0$, let $X := C([-\tau, 0]; \mathbb{R}^d)$ and $F : X \to \mathbb{R}^d$. A **Retarded Functional Differential Equation (RFDE)** is a relation

$$x'(t) = F(x_t),$$

where $x_t \in X$ is defined as

$$x_t(\theta) := x(t + \theta), \ \theta \in [-\tau, 0].$$

**Usual terminology:**
- $d$ is the dimension
- $\tau$ is the (maximum) delay
- $[-\tau, 0]$ is the delay interval
- $x_t$ is the state at time $t$
- $X$ is the state space
- $F$ is the Right-Hand Side (RHS, in general nonlinear, here autonomous).

**Common acronym:** DDEs for Delay Differential Equations.
examples

- **linear systems with discrete delay(s):**

  \[ F(\psi) = A\psi(0) + B\psi(-\tau) \Rightarrow x'(t) = Ax(t) + Bx(t - \tau) \]

- **linear systems with distributed delay(s):**

  \[ F(\psi) = A\psi(0) + \int_{-\tau}^{0} C(\theta)\psi(\theta)\,d\theta \Rightarrow x'(t) = Ax(t) + \int_{-\tau}^{0} C(\theta)x(t+\theta)\,d\theta \]

- **nonlinear delay logistic equation [13]:**

  \[ F(\psi) = r\psi(0)[1 - \psi(-\tau)] \Rightarrow x'(t) = rx(t)[1 - x(t - \tau)] \]

- **nonlinear Mackey-Glass equation [14]:**

  \[ F(\psi) = \frac{a\psi(-\tau)}{1 + [\psi(-\tau)]^c} - b\psi(0) \Rightarrow x'(t) = \frac{ax(t - \tau)}{1 + [x(t - \tau)]^c} - bx(t) \]
DDEs vs ODEs

Contrary to ODEs for $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$\begin{cases} x'(t) = f(x(t)), & t \geq 0, \\ x(0) = v \in \mathbb{R}^d, \end{cases}$$

a function $\varphi \in X$ is necessary for a specific solution of a DDE:

$$\begin{cases} x'(t) = F(x_t), & t \geq 0, \\ x(\theta) = \varphi(\theta), & \theta \in [-\tau, 0]. \end{cases}$$

**Theorem.** $F$ Lipschitz $\Rightarrow \exists!$ solution and it is Lipschitz w.r.t. $\varphi$.

A big difference: **DDEs generate $\infty$-dimensional dynamical systems.**

Other differences:
- no backward continuation if $\varphi$ only continuous
- more smoothness of $F \Rightarrow$ more smoothness of $x$ necessarily
- oscillations and chaos already for $d = 1$ (e.g., Mackey-Glass).
basics [9,12]

dynamical systems [9,10]

three different views

numerical analysis [1,5,11,15]
local stability of equilibria

Consider a linear(ized) continuous RHS $L : X \rightarrow \mathbb{R}^d$:

$$x'(t) = Lx_t.$$  \hspace{1cm} (1)

Looking for $x(t) = e^{\lambda t}v$, $v \in \mathbb{R}^d \setminus \{0\}$, leads to the characteristic equation

$$\det(\lambda I_d - Le^{\lambda t}) = 0.$$ \hspace{1cm} (2)

Its solutions $\lambda \in \mathbb{C}$ are known as characteristic roots.

**Theorem.** The zero solution of (1) is asymptotically stable $\Leftrightarrow \Re(\lambda) < 0$ for all $\lambda$, unstable if $\Re(\lambda) > 0$ for some $\lambda$.

Contrary to ODEs, (2) is a nonlinear eigenvalue problem: $\infty$-many $\lambda$.

**Theorem.** $\exists \alpha \in \mathbb{R}$ s.t. $\Re(\lambda) < \alpha$ for all $\lambda$ and there are only finitely-many $\lambda$ in any vertical strip of $\mathbb{C}$.

There exists an alternative characterization, which we follow for computation.
solution operator

Well-posedness allows to define the solution operator \( T(t) : X \to X \) as

\[
T(t) \varphi = x_t, \ t \geq 0.
\]

It associates to the initial function \( \varphi \) in \([-\tau, 0]\) the piece of solution \( x \) in \([t - \tau, t]\), shifted back to \([-\tau, 0]\).

**Theorem.** \( \{T(t)\}_{t \geq 0} \) is a \( C_0 \)-semigroup of bounded linear operators:

(a) \( T(t) \) is linear and bounded for all \( t \geq 0 \)

(b) \( T(0) = I_X \)

(c) \( T(t + s) = T(t)T(s) \) for all \( t, s \geq 0 \)

(d) \( \{T(t)\}_{t \geq 0} \) is strongly continuous: \( \lim_{t \downarrow 0} T(t)\psi = \psi \) for all \( \psi \in X. \)

What above is general for evolution maps of dynamical systems. What below is not always the case, but it holds for DDEs.

**Theorem.** \( T(t) \) is compact for all \( t \geq \tau. \)
infinitesimal generator and ∞-ODE

$C_0$-semigroups have infinitesimal generators. Here follows that associated to

\[ x'(t) = Lx_t. \]

**Theorem.** The infinitesimal generator is the linear unbounded operator $A : \mathcal{D}(A) \subseteq X \rightarrow X$ with action

\[ A\psi = \psi' \]

and domain

\[ \mathcal{D}(A) = \{ \psi \in X : \psi' \in X \text{ and } \psi'(0) = L\psi \}. \]

**Theorem (abstract Cauchy problem).** $u(t) = T(t)\varphi = x_t$ solves

\[ \begin{cases} u'(t) = Au(t), \ t \geq 0, \\ u(0) = \varphi \in \mathcal{D}(A). \end{cases} \]

A linear DDE is an $\infty$-dimensional linear ODE, governing the time-evolution of the state $u(t) = x_t$ in $X$. 
spectra and stability

The eigenvalues of the matrix $A$ rule the dynamics of the linear ODE

\[ x'(t) = Ax(t) \quad (\text{in } \mathbb{R}^d). \]

The linear DDE

\[ x'(t) = Lx_t \quad (\text{in } \mathbb{R}^d) \]

can be seen as

\[ u'(t) = Au(t) \quad (\text{in } X). \]

Thanks to compactness, the dynamics depends on the spectrum $\sigma(A)$ of $A$.

**Theorem.** $A$ and $T(t)$ for $t \geq \tau$ have only point spectrum (eigenvalues) and

\[ \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}. \]

The characteristic equation and dynamical systems standpoints are the same:

**Theorem.** $\lambda$ is a characteristic root $\iff \lambda \in \sigma(A)$.

(but not numerically, like for ODEs: polynomial roots vs matrix eigenvalues).
basics [9,12]

dynamical systems [9,10]

three different views

numerical analysis [1,5,11,15]
\[
\begin{cases}
x'(t) = Lx_t, & t \geq 0, \\
x(\theta) = \varphi(\theta), & \theta \in [-\tau, 0].
\end{cases}
\]
DDE $\rightarrow \infty$-ODE
DDE $\rightarrow \infty$-ODE

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DDE $\rightarrow \infty$-ODE
DDE $\rightarrow \infty$-ODE

from DDEs to ODEs (through PDEs)
\[
\begin{aligned}
\inf\text{-ODE} \\
\begin{cases}
  u'(t) = Au(t), & t \geq 0, \\
  u(0) = \varphi \in D(A).
\end{cases}
\end{aligned}
\]
PDE formulation

So far (1) $x(t) \in \mathbb{R}^d$ solves the Cauchy problem for the DDE

$$\begin{cases}
x'(t) = Lx_t, & t \geq 0, \\
x(\theta) = \varphi(\theta), & \theta \in [-\tau, 0],
\end{cases}$$

and (2) $u(t) = x_t \in X$ solves the abstract Cauchy problem for the $\infty$-ODE

$$\begin{cases}
u'(t) = Au(t), & t \geq 0, \\
u(0) = \varphi \in D(A).
\end{cases}$$

Consider now

$$v(t, \theta) := x_t(\theta) = [u(t)](\theta).$$

Since $x_t(\theta) = x(t + \theta)$, we have

$$\frac{\partial x_t(\theta)}{\partial t} = \frac{\partial x(t + \theta)}{\partial t} = \frac{\partial x(t + \theta)}{\partial \theta} = \frac{\partial x_t(\theta)}{\partial \theta}.$$

Hence (3) $v(t, \theta) \in \mathbb{R}^d$ solves the initial-boundary value problem for the PDE

$$\begin{cases}
\frac{\partial v(t, \theta)}{\partial t} = \frac{\partial v(t, \theta)}{\partial \theta}, & t \geq 0, \ \theta \in [-\tau, 0], \\
\frac{\partial v(t, \theta)}{\partial t} \bigg|_{\theta=0} = Lv(t, \cdot), & t \geq 0, \\
v(0, \theta) = \varphi(\theta), & \theta \in [-\tau, 0].
\end{cases}$$
\begin{align*}
\begin{cases}
    u'(t) &= Au(t), \quad t \geq 0, \\
    u(0) &= \varphi \in \mathcal{D}(A).
\end{cases}
\end{align*}
\[
\begin{aligned}
        x'(t) &= Lx_t, \quad t \geq 0, \\
        x(\theta) &= \varphi(\theta), \quad \theta \in [-\tau, 0].
    \end{aligned}
\]
DDE → PDE

\[
\frac{\partial v(t, \theta)}{\partial t} = \frac{\partial v(t, \theta)}{\partial \theta}
\]

characteristic lines \( t + \theta = \text{constant} \).
DDE → PDE

from DDEs to ODEs (through PDEs)

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DDE $\rightarrow$ PDE

from DDEs to ODEs (through PDEs)
\[ \begin{align*}
\frac{\partial v(t, \theta)}{\partial t} &= \frac{\partial v(t, \theta)}{\partial \theta}, & t \geq 0, \theta \in [-\tau, 0], \\
\left. \frac{\partial v(t, \theta)}{\partial t} \right|_{\theta=0} &= Lv(t, \cdot), & t \geq 0, \\
v(0, \theta) &= \varphi(\theta), & \theta \in [-\tau, 0].
\end{align*} \]
nonlinear case

For nonlinear $F$, the DDE
\[
\begin{aligned}
    x'(t) &= F(x_t), \ t \geq 0, \\
    x(\theta) &= \varphi(\theta), \ \theta \in [-\tau, 0],
\end{aligned}
\]
is still equivalent to the PDE
\[
\begin{aligned}
    \frac{\partial v(t, \theta)}{\partial t} &= \frac{\partial v(t, \theta)}{\partial \theta}, \quad t \geq 0, \ \theta \in [-\tau, 0], \\
    \frac{\partial v(t, \theta)}{\partial t} \bigg|_{\theta = 0} &= F(v(t, \cdot)), \ t \geq 0, \\
    v(0, \theta) &= \varphi(\theta), \quad \theta \in [-\tau, 0].
\end{aligned}
\]

The abstract Cauchy problem reads
\[
\begin{aligned}
    u'(t) &= A(u(t)), \ t \geq 0, \\
    u(0) &= \varphi \in D(A),
\end{aligned}
\]
with $A$ the **nonlinear** differential operator
\[
A\psi = \psi' \quad \text{and} \quad D(A) = \{ \psi \in X : \ \psi' \in X \ \text{and} \ \psi'(0) = F(\psi) \}.
\]

From now on let $d = 1$ without loss of generality and remember that
\[
v(t, \theta) = x_t(\theta) = [u(t)](\theta).
\]
basics [9,12]

dynamical systems [9,10]

three different views

numerical analysis [1,5,11,15]
polynomial interpolation

Let a function \( f : [-1, 1] \rightarrow \mathbb{R} \) be known on given nodes \( t_i, \ i = 0, 1, \ldots, N \). Polynomial interpolation is a way to approximate \( f \): find \( p_N \in \Pi_N \) s.t.

\[
p_N(t_i) = f(t_i).
\]

**Theorem.** \( \exists! p_N \in \Pi_N \) interpolating \( f \) for any choice of \( N + 1 \) distinct nodes.

There are several ways to represent \( p_N \). We use the Lagrange form:

\[
p_N(t) = \sum_{j=0}^{N} \ell_j(t)f(t_j),
\]

where

\[
\ell_j(t) = \prod_{\substack{k=0 \atop k \neq j}}^{N} \frac{t - t_k}{t_j - t_k}, \ t \in [-1, 1].
\]

**Proof:** \( p_N(t_i) = f(t_i) \) since \( \ell_j(t_i) = \delta_{i,j} \), the Kronecker's delta. \( \square \)

Interpolation is fine for smooth \( f \) and “good” nodes, as Chebyshev nodes are:

\[
t_i = \cos \left( \frac{i\pi}{N} \right).
\]
pseudospectral approximation

PSA: whatever you do with \( f \), do it instead with \( p_N \) (dimension: \( \infty \to \text{finite} \)).

Example on differentiation. Assume we know

\[
y_i = f(t_i)
\]

for \( i = 0, 1, \ldots, N \). Set \( y = (y_0, y_1, \ldots, y_N)^T \). Let \( z = (z_0, z_1, \ldots, z_N)^T \) with

\[
z_i = p'_N(t_i)
\]

approximating \( f'(t_i) \). Differentiation and interpolation being linear, we write

\[
z = D_Ny,
\]

where \( D_N \in \mathbb{R}^{(N+1) \times (N+1)} \) is the differentiation matrix

\[
D_N = \begin{pmatrix}
d_{0,0} & d_{0,1} & \cdots & d_{0,N} \\
d_{1,0} & d_{1,1} & \cdots & d_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
d_{N,0} & d_{N,1} & \cdots & d_{N,N}
\end{pmatrix} = \begin{pmatrix}
\ell'_0(t_0) & \ell'_1(t_0) & \cdots & \ell'_N(t_0) \\
\ell'_0(t_1) & \ell'_1(t_1) & \cdots & \ell'_N(t_1) \\
\vdots & \vdots & \ddots & \vdots \\
\ell'_0(t_N) & \ell'_1(t_N) & \cdots & \ell'_N(t_N)
\end{pmatrix}.
\]
**Example on differentiation.** Assume we know

\[
y_i = f(t_i)
\]

for \( i = 0, 1, \ldots, N \). Set \( y = (y_0, y_1, \ldots, y_N)^T \). Let \( z = (z_0, z_1, \ldots, z_N)^T \) with

\[
z_i = p'_N(t_i)
\]

approximating \( f'(t_i) \). Differentiation and interpolation being linear, we write

\[
z = D_N y,
\]

where \( D_N \in \mathbb{R}^{(N+1) \times (N+1)} \) is the differentiation matrix

\[
D_N = \begin{pmatrix}
d_{0,0} & d_{0,1} & \cdots & d_{0,N} \\
d_{1,0} & d_{1,1} & \cdots & d_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
d_{N,0} & d_{N,1} & \cdots & d_{N,N} \\
\tilde{d}_N & \tilde{d}_N & & \\
\end{pmatrix}
\]

\[
\tilde{D}_N = \begin{pmatrix}
\ell'_0(t_0) & \ell'_1(t_0) & \cdots & \ell'_N(t_0) \\
\ell'_0(t_1) & \ell'_1(t_1) & \cdots & \ell'_N(t_1) \\
\vdots & \vdots & \ddots & \vdots \\
\ell'_0(t_N) & \ell'_1(t_N) & \cdots & \ell'_N(t_N) \\
\end{pmatrix}
\]
spectral accuracy

Compare PSA on Chebyshev nodes (red) with finite differences (blue)

\[ f'(t) \approx \frac{f(t+h) - f(t)}{h}, \quad h = \frac{2}{N}, \]

to approximate \( f' \) for \( f(t) = e^{-t^2} \).

PSA exploits all the smoothness of \( f: \infty \) vs finite convergence order.

For Chebyshev nodes, moreover, \( D_N \) is known explicitly.
The idea is to reduce $\mathcal{A}$ to a matrix and use its eigenvalues as approximations. Recall that

$$\mathcal{A}\psi = \psi' \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = \{\psi \in X : \psi' \in X \text{ and } \psi'(0) = L\psi\}.$$ 

Let $p_N$ interpolate $\psi \in X$ on the Chebyshev nodes in $[-\tau, 0]$

$$\theta_i = \frac{\tau}{2} \cos \left(\frac{i\pi}{N}\right) - \frac{\tau}{2}, \ i = 0, 1, \ldots, N.$$ 

Notice that $\theta_0 = 0$ and $\theta_N = -\tau$.

Then

$$(\mathcal{A}\psi)(\theta_i) = \psi'(\theta_i) \approx p_N'(\theta_i)$$

for $i = 1, \ldots, N$ while at $\theta_0 = 0 \ (i = 0)$ we impose

$$(\mathcal{A}\psi)(0) = \psi'(0) = L\psi \approx Lp_N$$

to respect the condition in $\mathcal{D}(\mathcal{A})$. 
discretized infinitesimal generator

Being $L$, differentiation and interpolation linear, the relation between the values $\psi(\theta_i)$ and the approximations of $(A\psi)(\theta_i)$ is given by a matrix. Indeed, if

$$\psi(\theta) \approx p_N(\theta) = \sum_{j=0}^{N} \ell_j(\theta)\psi(\theta_j),$$

then for $i = 0$

$$(A\psi)(0) \approx Lp_N = \sum_{j=0}^{N} L\ell_j\psi(\theta_j)$$

while for $i = 1, \ldots, N$

$$(A\psi)(\theta_i) \approx p'_N(\theta_i) = \sum_{j=0}^{N} d_{i,j}\psi(\theta_j).$$

Therefore $A$ is discretized by

$$A_N = \begin{pmatrix} L\ell_0 & L\ell_1 & \cdots & L\ell_N \\ d_{1,0} & d_{1,1} & \cdots & d_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N,0} & d_{N,1} & \cdots & d_{N,N} \end{pmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}.$$
discretized infinitesimal generator

Being \( L \), differentiation and interpolation linear, the relation between the values \( \psi(\theta_i) \) and the approximations of \( (A\psi)(\theta_i) \) is given by a matrix. Indeed, if

\[
\psi(\theta) \approx p_N(\theta) = \sum_{j=0}^{N} \ell_j(\theta)\psi(\theta_j),
\]

then for \( i = 0 \)

\[
(A\psi)(0) \approx Lp_N = \sum_{j=0}^{N} L\ell_j\psi(\theta_j)
\]

while for \( i = 1, \ldots, N \)

\[
(A\psi)(\theta_i) \approx p'_N(\theta_i) = \sum_{j=0}^{N} d_{i,j}\psi(\theta_j).
\]

Therefore \( A \) is discretized by

\[
A_N = \begin{pmatrix}
L\ell_0 & L\ell_1 & \cdots & L\ell_N \\
\text{d}_{1,0} & \text{d}_{1,1} & \cdots & \text{d}_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
\text{d}_{N,0} & \text{d}_{N,1} & \cdots & \text{d}_{N,N}
\end{pmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}.
\]
example

Take
\[ L\psi = a\psi(0) + b\psi(-\tau) \implies x'(t) = ax(t) + bx(t - \tau). \]

Since \( \theta_0 = 0, \theta_N = -\tau \) and \( \ell_j(\theta_i) = \delta_{i,j} \) we get

\[
\begin{align*}
L\ell_0 &= a, \\
L\ell_j &= 0, \quad j = 1, \ldots, N - 1, \\
L\ell_N &= b.
\end{align*}
\]

Therefore
\[
A_N = \begin{pmatrix}
a & 0 & \cdots & 0 & b \\
d_{1,0} & d_{1,1} & \cdots & d_{1,N-1} & d_{1,N} \\
& & & & \\
& & & & \\
d_{N,0} & d_{N,1} & \cdots & d_{N,N-1} & d_{N,N}
\end{pmatrix}.
\]

Observe: only the first row of \( A_N \) is influenced by the RHS of the DDE, the rest depends only on \( \tau \) and on the Chebyshev nodes in \([-\tau, 0]\).
Theorem. Let \( \lambda \) be an eigenvalue of \( A \) with multiplicity \( m \). For \( N \) sufficiently large, \( A_N \) has exactly \( m \) eigenvalues \( \lambda_i \) counted with multiplicities and

\[
\max_{i=1, \ldots, m} |\lambda - \lambda_i| \leq C_0 \left( \frac{C_1}{N} \right)^{N/m}
\]

with \( C_0, C_1 \) constants and \( C_1 \) proportional to \( |\lambda| \).
extension to the nonlinear case

PSA works well for the local stability analysis of equilibria through the computation of the eigenvalues of $A$.

It is just the first step in the dynamical analysis. Next steps are stability of periodic orbits (Floquet multipliers) and detection of chaotic behaviors (Lyapunov exponents). Other directions concern with bifurcation analysis or different classes of functional equations (neutral and state-dependent, retarded-advanced, retarded partial, differential-algebraic, integro-differential, etc.).

For most of the above, PSA is well-suited [2,3,4,6].
extension to the nonlinear case

PSA works well for the local stability analysis of equilibria through the computation of the eigenvalues of $A$.

It is just the first step in the dynamical analysis. Next steps are stability of periodic orbits (Floquet multipliers) and detection of chaotic behaviors (Lyapunov exponents). Other directions concern with bifurcation analysis or different classes of functional equations (neutral and state-dependent, retarded-advanced, retarded partial, differential-algebraic, integro-differential, etc.).

For most of the above, PSA is well-suited [2,3,4,6].

What about the nonlinear case?

There is more behind the nature of PSA. Especially related to the following question: if DDEs are $\infty$-ODEs, what do we loose or preserve by considering just a finite number of the latter?

Observe that the interest is not in approximating a specific solution, but rather the whole dynamics.
PSA of nonlinear DDEs

Consider

\begin{align*}
\begin{cases}
x'(t) &= F(x_t), \quad t \geq 0, \\
x(\theta) &= \phi(\theta), \quad \theta \in [-\tau, 0].
\end{cases}
\end{align*}

For \( t \geq 0 \) and \(-\tau = \theta_N < \cdots < \theta_0 = 0\) the Chebyshev nodes in \([-\tau, 0]\), let

\[ \tilde{u}_i(t) \approx x_t(\theta_i) \]

be an approximation in \( \theta_i \) of the state at time \( t \). Their interpolant

\[ \tilde{v}(t, \theta) = \sum_{j=0}^{N} \ell_j(\theta)\tilde{u}_j(t) \]

is a polynomial in \( \theta \in [-\tau, 0] \), yet a function of \( t \geq 0 \) since so are the \( \tilde{u}_i \)'s.

**Question:** does \( \tilde{v} \) satisfy a differential equation and, if so, what is the relation with the original DDE?
approximated PDE

It is not difficult to prove that

\[
\bar{v}(t, \theta) = \sum_{j=0}^{N} \ell_j(\theta) \bar{u}_j(t)
\]

satisfies

\[
\begin{align*}
\frac{\partial \bar{v}(t, \theta)}{\partial t} &= \frac{\partial \bar{v}(t, \theta)}{\partial \theta} + \epsilon(t, \theta, \bar{v}), \quad t \geq 0, \ \theta \in [-\tau, 0], \\
\frac{\partial \bar{v}(t, \theta)}{\partial t} \bigg|_{\theta=0} &= F(\bar{v}(t, \cdot)), \quad t \geq 0, \\
\bar{v}(0, \theta) &= \sum_{j=0}^{N} \ell_j(\theta) \varphi(\theta_j), \quad \theta \in [-\tau, 0].
\end{align*}
\]

It corresponds to the PDE formulation of the DDE, but for the interpolation of the initial function \( \varphi \) and the error term \( \epsilon(t, \theta, \bar{v}) \), which we see in a moment.

So \( \bar{v}(t, \theta) \) approximates \( v(t, \theta) \)...and \( \bar{u}_j(t) \)?
The error term is
\[ \varepsilon(t, \theta, \bar{v}) = \ell_0(\theta) \left( F(\bar{v}(t, \cdot)) - \frac{\partial \bar{v}(t, \theta)}{\partial \theta} \bigg|_{\theta=0} \right) \]

and it vanishes at \( \theta_i \) for \( i = 1, \ldots, N \) since \( \ell_j(\theta_i) = \delta_{i,j} \).

Therefore \( \bar{v}(t, \cdot) \) is the polynomial of collocation of the original PDE:

\[
\begin{align*}
\frac{\partial \bar{v}(t, \theta)}{\partial t} \bigg|_{\theta=\theta_i} &= \frac{\partial \bar{v}(t, \theta)}{\partial \theta} \bigg|_{\theta=\theta_i}, & i = 1, \ldots, N, \\
\frac{\partial \bar{v}(t, \theta)}{\partial t} \bigg|_{\theta=0} &= F(\bar{v}(t, \cdot)), & (i = 0), \\
\bar{v}(0, \theta_i) &= \varphi(\theta_i), & i = 0, 1, \ldots, N.
\end{align*}
\]

But this is equivalent to the Cauchy problem

\[
\begin{align*}
\bar{u}_0'(t) &= F(\bar{v}(t, \cdot)), & (i = 0), \ t \geq 0 \\
\bar{u}_i'(t) &= \sum_{j=0}^{N} d_{i,j} \bar{u}_j(t), & i = 1, \ldots, N, \ t \geq 0 \\
\bar{u}_i(0) &= \varphi(\theta_i), & i = 0, 1, \ldots, N,
\end{align*}
\]

for a system of \( N + 1 \) ODEs, nonlinear only in the first one.
If $F = L$ is linear, the first ODE becomes

$$\ddot{u}_0'(t) = L\ddot{v}(t, \cdot) = \sum_{j=0}^{N} L\ell_j \ddot{u}_j(t).$$

Therefore $\ddot{u}(t) = (\ddot{u}_0(t), \ddot{u}_1(t), \ldots, \ddot{u}_N(t))^T$ solves

$$\begin{cases}
\ddot{u}'(t) = A_N \ddot{u}(t), \\
\ddot{u}(0) = \phi \in \mathbb{R}^{N+1},
\end{cases}$$

for $\phi = (\phi(\theta_0), \phi(\theta_1), \ldots, \phi(\theta_N))^T$. Exactly the discretization of the original abstract Cauchy problem

$$\begin{cases}
u'(t) = Au(t), \\
u(0) = \phi \in X.
\end{cases}$$
\[ \begin{align*} 
\frac{\partial v(t, \theta)}{\partial t} &= \frac{\partial v(t, \theta)}{\partial \theta}, \quad t \geq 0, \ \theta \in [-\tau, 0], \\
\left. \frac{\partial v(t, \theta)}{\partial t} \right|_{\theta=0} &= F(v(t, \cdot)), \quad t \geq 0, \\
v(0, \theta) &= \varphi(\theta), \quad \theta \in [-\tau, 0]. 
\end{align*} \]
PDE $\rightarrow N + 1$ ODEs

$\phi = (\varphi(\theta_0), \varphi(\theta_1), \ldots, \varphi(\theta_N))^T$
N + 1 ODEs

\[
\begin{aligned}
\bar{u}'_0(t) &= F(\bar{v}(t, \cdot)), & (i = 0), \ t \geq 0 \\
\bar{u}'_i(t) &= \sum_{j=0}^{N} d_{i,j} \bar{u}_j(t), & i = 1, \ldots, N, \ t \geq 0 \\
\bar{u}_i(0) &= \varphi(\theta_i), & i = 0, 1, \ldots, N.
\end{aligned}
\]
Consider the linear part

\[ \tilde{u}_i'(t) = \sum_{j=0}^{N} d_{i,j} \tilde{u}_j(t), \quad i = 1, \ldots, N, \]

and recall that

\[ \tilde{d}_N = \begin{pmatrix} d_{1,0} \\ \vdots \\ d_{N,0} \end{pmatrix}, \quad \tilde{D}_N = \begin{pmatrix} d_{1,1} & \cdots & d_{1,N} \\ \vdots & \ddots & \vdots \\ d_{N,1} & \cdots & d_{N,N} \end{pmatrix}. \]

Then \( \tilde{u}(t) = (\tilde{u}_1(t), \ldots, \tilde{u}_N(t))^T \) satisfies the linear inhomogeneous ODE

\[ \tilde{u}'(t) = \tilde{D}_N \tilde{u}(t) + \tilde{d}_N \tilde{u}_0(t), \]

whose free dynamics depends on the eigenvalues of \( \tilde{D}_N \).

\( \tilde{D}_N \) is independent of the RHS of the DDE, so we study the easiest case \( F \equiv 0 \):

\[
\begin{cases}
  x'(t) = 0, & t \geq 0, \\
  x(\theta) = \varphi(\theta), & \theta \in [-\tau, 0].
\end{cases}
\]

It has constant solution \( x(t) = \varphi(0) \). Let \( \varphi(0) = 0 \) without loss of generality.
properties of $\sigma(\tilde{D}_N)$

The first ODE gives $\tilde{u}_0(t) = \varphi(0) = 0$ and the linear part becomes

$$\tilde{u}'(t) = \tilde{D}_N \tilde{u}(t).$$ (1)

Also $\tilde{u}(t)$ for $t \geq \tau$ must vanish as $N \to \infty$ to converge to the exact solution.

**Theorem.** $\tilde{D}_N$ is nonsingular.

**Proof:** $e^{\lambda t} \tilde{w}$ solves (1) for $\lambda \in \sigma(\tilde{D}_N)$ with eigenvector $\tilde{w}$. The interpolant

$$\tilde{v}(t, \theta) = \ell_0(\theta) \tilde{u}_0(t) + \sum_{j=1}^{N} \ell_j(\theta) e^{\lambda t} w_j = q(\theta) e^{\lambda t} \quad (q \in \Pi_N)$$

solves the approximated PDE, hence

$$q'(\theta) = \lambda q(\theta) + \ell_0(\theta) q'(0).$$

Finally, $\ell_0 \in \Pi_N$ and $q' \in \Pi_{N-1}$ imply $\lambda \neq 0$. □

**Theorem.** $\sigma(\tilde{D}_N) \subset \mathbb{C}^-$ for all positive integers $N$.

**Conjecture.** $\Re(\sigma(\tilde{D}_N)) \to -\infty$ as $N \to \infty$. 

from DDEs to ODEs (through PDEs) dynamical systems and applications @ BCAM 32/40
Theorem. A constant function of value $c \in \mathbb{R}$ is an equilibrium of the DDE $x'(t) = F(x_t)$ iff the constant vector $(c, \ldots, c)^T \in \mathbb{R}^{N+1}$ is an equilibrium of the ODE approximation

$$\begin{cases} \bar{u}_0'(t) = F(\bar{v}(t, \cdot)), \\ \tilde{u}'(t) = \tilde{D}_N \tilde{u}(t) + \tilde{d}_N \bar{u}_0(t). \end{cases}$$

Proof: based on two properties of Lagrange interpolation:

$$\sum_{j=0}^{N} \ell_j(\theta) = 1 \quad \forall \theta \in [-\tau, 0] \quad \text{and} \quad \sum_{j=0}^{N} d_{i,j} = 0 \quad \forall i = 0, 1, \ldots, N.$$ 

The second for $i = 1, \ldots, N$ reads $\tilde{D}_N (1, \ldots, 1)^T + \tilde{d}_N = 0$. 

Theorem. Linearization and approximation commute.

Proof: the only nonlinear part of the ODE is the RHS $F$. 

Theorem. The linearized ODE approximation predicts accurately the local stability properties of the equilibria of the nonlinear DDE.

Proof: it gives the discretized infinitesimal generator.
bifurcation of equilibria

Concluding, the bifurcation analysis of equilibria of nonlinear DDEs can be tackled efficiently by standard tools for ODEs (e.g., AUTO [16], MATCONT [17]).

A simple test. The nonlinear logistic DDE

\[ x'(t) = rx(t)[1 - x(t - 1)] \]

has equilibrium \( c = 1 \) for all \( r \): asymptotically stable for \( r \in [0, \pi/2) \), unstable otherwise. At \( r = \pi/2 \) a Hopf bifurcation occurs and a limit cycle arises.
more on

The same should apply to periodic orbits: those of the ODE approximation are spectrally accurate approximations of the exact ones; linearization and approximation always commute; PSA of Floquet multipliers is available [3].

Ongoing work to provide theoretical support:

- $\Re(\sigma(\tilde{D}_N)) \to -\infty$ as $N \to \infty$: how fast?
- Convergence of $\bar{v}(t, \theta)$ to $\chi_t(\theta)$: spectrally accurate for smooth $x$?
more on

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Ongoing work to provide theoretical support:

- \( \Re(\sigma(\tilde{D}_N)) \to -\infty \) as \( N \to \infty \): how fast?
- convergence of \( \tilde{v}(t, \theta) \) to \( x_t(\theta) \): spectrally accurate for smooth \( x \)?

The general idea (and hope) is that standard ODE tools as applied to the ODE approximation can easily provide information on the dynamics of the original DDE. Advantages:

- valid for any DDE
- bifurcation tools for ODEs complete and efficient; not so for DDEs [18]
- spectral accuracy (low \( N \))
- theoretical insight (dynamical systems, functional analysis, \( \odot \star [9] \)).

from DDEs to ODEs (through PDEs)

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the true (long-term) objective

Recent sophisticated models of resource-consumer dynamics are based on delay integro-differential equations (DEs/DDEs) such as [8]

\[
\begin{align*}
\frac{b(t)}{h} &= \int_{0}^{h} \beta(X(a, S_t), S(t))F(a, S_t)b(t - a)\,da, \\
\frac{S'(t)}{f(S(t))} &= \int_{0}^{h} \gamma(X(a, S_t), S(t))F(a, S_t)b(t - a)\,da,
\end{align*}
\]

with \(X\) and \(F\) solutions of external ODEs and discontinuities between juveniles and adults.

Semigroup and stability theories are available [7], as well as the PSA of eigenvalues, but only for caricature models [2].

The ODE approximation presented for DDEs is adaptable to such models.

Ongoing work:
- computation of eigenvalues (linear);
- bifurcation analysis (nonlinear).
research group

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...thanks for your attention!


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