Fractal properties of generalized Bessel functions

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Overview of the presentation

- We analyze fractal properties of oscillatory solutions of

$$t^2x''(t) + t(2-\mu)x'(t) + (t^2 - \nu^2)x(t) = 0, \tag{1}$$

  having parameters $\mu \in (0,2)$ and $\nu \in \mathbf{R}$ (it is *Bessel equation* for $\mu = 1$).
- We use the concept of *fractal dimension*, examining a *phase portrait* of solutions.
- We use Minkowski–Bouligand dimension, known as *box-counting dimension*.
- Phase portraits of solutions of (1) are *spirals* in the plane near the origin.
- Finally, we present some results of D. Žubrinić and V. Županović (2008), about the box dimension of focus spiral trajectories of *planar vector fields*

Minkowski content

Definition (ε-neighborhood)

Let $A \subset \mathbb{R}^n$, A is bounded. The **ε-neighborhood** *of set A is*

$$A_\varepsilon := \{y \in \mathbb{R}^n : d(y, A) < \varepsilon\}.$$

Definition (Lower an upper s-dimensional Minkowski content)

Lower s-dimensional Minkowski content *of bounded set $A \subset \mathbb{R}^n$, $s \geq 0$ is*

$$\mathscr{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{n-s}}.$$

Upper s-dimensional Minkowski content *$\mathscr{M}^{*s}(A)$, $s \geq 0$ is*

$$\mathscr{M}^{*s}(A) := \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{n-s}}.$$

If $\mathscr{M}^{*s}(A) = \mathscr{M}_*^s(A)$, the common value is called the **s-dimensional Minkowski content of** *A, and is denoted by $\mathscr{M}^s(A)$.*

Box dimension

Definition (Lower and upper box dimension)

Lower box dimension *of bounded set* $A \subset \mathbb{R}$ *is*

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathscr{M}_*^s(A) = 0\} = \sup\{s \geq 0 : \mathscr{M}_*^s(A) = \infty\}.$$

Upper box dimension *of* A *is*

$$\overline{\dim}_B A := \inf\{s \geq 0 : \mathscr{M}^{*s}(A) = 0\} = \sup\{s \geq 0 : \mathscr{M}^{*s}(A) = \infty\}.$$

Generally $\underline{\dim}_B A \leq \overline{\dim}_B A$.

Definition (Box dimension)

If $\underline{\dim}_B A = \overline{\dim}_B A$ *we define the* **box dimension** *of* A *to be*

$$\dim_B A := \underline{\dim}_B A = \overline{\dim}_B A.$$

Box dimension - examples

Examples of some sets and their box dimensions

Let $n = 2$, ambient space is $\mathbb{R}^2$.

- A is a single point, $\dim_B A = 0$
- A is a line segment, $\dim_B A = 1$
- A is a disk, $\dim_B A = 2$
- A is a smooth rectifiable curve, $\dim_B A = 1$
- A is a non-rectifiable power spiral, $0 < \alpha < 1$, given in polar coordinates by

$$r = \varphi^{-\alpha}, \quad \varphi \in [\varphi_0, \infty), \qquad \dim_B A = \frac{2}{1+\alpha} \in (1,2), \qquad \text{(C. Tricot, 1993)}$$

- A is a graph of a non-rectifiable (α, β)-chirp near the origin, $0 < \alpha < \beta$, given by

$$f(\tau) = \tau^\alpha \cos \tau^{-\beta} \quad \tau \in (0, \tau_0], \qquad \dim_B A = 2 - \frac{\alpha+1}{\beta+1} \in (1,2),$$

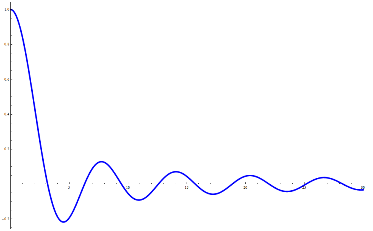
(C. Tricot, *Curves and Fractal Dimension*, 1993)

Oscillatory function near infinity

Definition (Oscillatory function near $t = \infty$)

Let $f : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, be a continuous function. $f(t)$ is an **oscillatory function** *near $t = \infty$ if there exists sequence $t_k \to \infty$ such that $f(t_k) = 0$, and restrictions $f|_{(t_k, t_{k+1})}$ intermittently change sign for $k \in \mathbb{N}$.*

Example: $f(x) = \frac{1}{x}\sin(x)$.
It is (α, β)-chirp near infinity, $f(t) = t^{-\alpha} \sin t^{\beta}$, $t \in [t_0, \infty)$, for $\alpha = \beta = 1$.

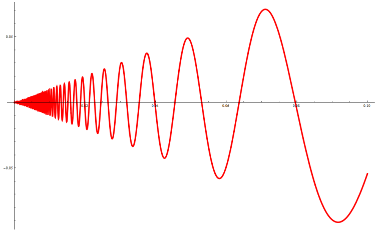


Oscillatory function near the origin

Definition (Oscillatory function near the origin)

Let $g : (0, \tau_0] \to \mathbb{R}$, $\tau_0 > 0$, *be a continuous function.* $g(t)$ *is an* **oscillatory function** *near the origin if there exists sequence* s_k *such that* $s_k \searrow 0$ *as* $k \to \infty$, $g(s_k) = 0$ *and restrictions* $g|_{(s_{k+1}, s_k)}$ *intermittently change sign for* $k \in \mathbb{N}$.

Example: $g(x) = x \sin(1/x)$.
Notice it is $(1,1)$-chirp near the origin.



Phase oscillatory function

Definition (Phase oscillatory function (Pašić, Žubrinić, Županović))

Let $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$ *and* $x \in C^1$. $x(t)$ *is a* **phase oscillatory** *function if set*

$$\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$$

in the plane is a spiral converging to the origin.

Definition (Spiral)

A **spiral** *is the graph of function* $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, *in polar coordinates, where*

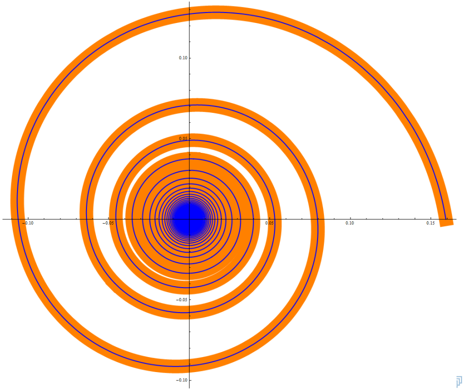
- $f : [\varphi_1, \infty) \to (0, \infty)$ *is such that* $f(\varphi) \to 0$ *as* $\varphi \to \infty$,
- f *is radially decreasing (ie, for any fixed* $\varphi \geq \varphi_1$ *the function* $\mathbb{N} \ni k \mapsto f(\varphi + 2k\pi)$ *is decreasing).*

A mirror image of a spiral over the x-axes will be also called a spiral.

Phase dimension

Definition (Phase dimension (Pašić, Žubrinić, Županović))

The **phase dimension** $\dim_{ph}(x)$ *of phase oscillatory function* $x(t)$ *is the box dimension of corresponding spiral* $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.

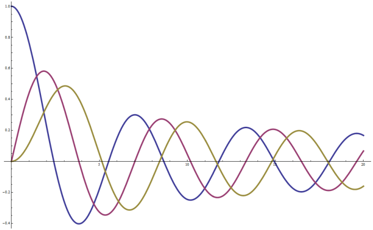


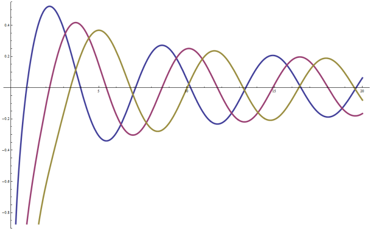
Bessel equation

Bessel equation of order ν

$$t^2x''(t) + tx'(t) + (t^2 - \nu^2)x(t) = 0, \quad \nu \in \mathbf{R}$$

- Linear second-order ordinary differential equation. Single parameter ν.
- Two linearly independent solutions are called Bessel functions $J_\nu(t)$ and $Y_\nu(t)$.
- The solutions are oscillatory functions near $t = \infty$.





Phase dimension of Bessel functions

Bessel system of order ν - substitution $y = \dot{x}$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\frac{\nu^2}{t^2} - 1)x - \frac{1}{t}y, \quad \nu \in \mathbf{R} \end{aligned}$$

Theorem (Phase dimension of Bessel functions (L. Korkut, D. Vlah, V. Županović))

The phase dimension of $J_\nu(t)$ and $Y_\nu(t)$ is

$$\dim_{ph}(J_\nu) = \dim_{ph}(Y_\nu) = \frac{4}{3}, \quad \textit{for all } \nu \in \mathbf{R}$$

Remark

Problem: A spiral radius function is **non-monotone** - we get a **wavy spiral**.

Reflected Bessel equation

Substitution $t = \frac{1}{\tau}$ in Bessel equation:

Reflected Bessel equation

$$x''(\tau) + \frac{1}{\tau} x'(\tau) + \left(\frac{1}{\tau^4} - \frac{\nu^2}{\tau^2} \right) x(\tau) = 0, \quad \nu \in \mathbf{R}$$

- Solutions are oscillatory near the origin.

Generalized reflected Bessel equation

$$x''(\tau) + \frac{\mu}{\tau} x'(\tau) + \left(\frac{\lambda}{\tau^\sigma} - \frac{\nu^2}{\tau^2} \right) x(\tau) = 0, \quad \mu \in \mathbf{R},\ \lambda > 0,\ \sigma > 2,\ \nu \in \mathbf{R}$$

- Introduced by Pašić, Tanaka, 2011.
- Solutions are oscillatory near the origin.
- They determined box dimension of graphs of solutions - oscillatory dimension

Back to ∞

Substitution $\tau = \frac{1}{t}$ will get us back to ∞.

Generalized Bessel equation

$$t^2 x''(t) + t(2-\mu)x'(t) + (\lambda t^{\sigma-2} - \nu^2)x(t) = 0, \quad \mu \in \mathbf{R},\ \lambda > 0,\ \sigma > 2,\ \nu \in \mathbf{R}$$

- Two linearly independent solutions.
- Solutions are oscillatory near $t = \infty$.
- We would like to determine **phase dimension** of the solutions.
- $\mu = 1$, $\lambda = 1$, $\sigma = 4$, $\nu \in \mathbf{R}$ is the standard Bessel equation.
- We fix $\lambda = 1$, $\sigma = 4$.
- What is the phase dimension of the solutions depending on parameters μ and ν?

Phase dimension of solutions of generalized Bessel equation

Generalized Bessel equation for $\mu \in \mathbf{R}$, $\lambda = 1$, $\sigma = 4$ and $\nu \in \mathbf{R}$

$$t^2 x''(t) + t(2-\mu)x'(t) + (t^2 - \nu^2)x(t) = 0$$

Two linearly independent solutions we call **generalized Bessel functions**

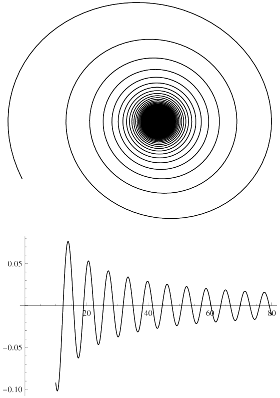
$$\begin{aligned} \widetilde{J}_{\nu,\mu}(t) &= t^{\frac{\mu-1}{2}} J_{\widetilde{\nu}}(t), \\ \widetilde{Y}_{\nu,\mu}(t) &= t^{\frac{\mu-1}{2}} Y_{\widetilde{\nu}}(t), \quad \text{where } \widetilde{\nu} = \sqrt{\left(\frac{\mu-1}{2}\right)^2 + \nu^2}. \end{aligned}$$

Theorem (Phase dimension of generalized Bessel functions (L. Korkut, D. Vlah, V. Županović))

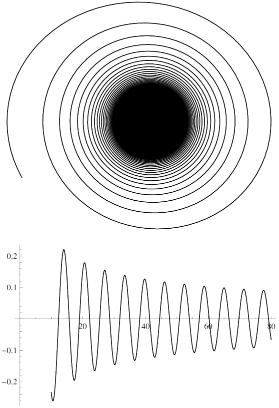
The phase dimension of $\widetilde{J}_{\nu,\mu}(t)$ and $\widetilde{Y}_{\nu,\mu}(t)$ is

$$\dim_{ph}(\widetilde{J}_{\nu,\mu}) = \dim_{ph}(\widetilde{Y}_{\nu,\mu}) = \frac{4}{4-\mu}, \quad \textit{for all } \mu \in (0,2),\ \nu \in \mathbf{R}.$$

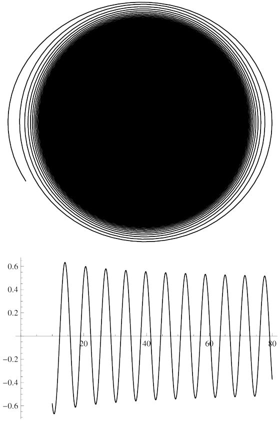
Generalized Bessel functions



$x_1(t) = \widetilde{J}_{5,0.2}(t)$,
$\dim_{ph}(x_1) = \frac{20}{19}$



$x_2(t) = \widetilde{J}_{5,1}(t)$,
$\dim_{ph}(x_2) = \frac{4}{3}$



$x_3(t) = \widetilde{J}_{5,1.8}(t)$,
$\dim_{ph}(x_3) = \frac{20}{11}$

Box dimension of spiral solutions of a planar system

Theorem (D. Žubrinić, V. Županović (2008))

Let Γ *be a spiral trajectory of a planar vector field of class* C^1. *Let* $P_\sigma(s)$ *be the Poincaré map with respect to an axis* σ, *and assume that it has the form* $P_\sigma(s) = s + d_\sigma(s)$ *for each* σ, *where the displacement function* $d_\sigma(\cdot) : (0, r_\sigma) \to (-\infty, 0)$ *is monotonically nonincreasing, such that* $-d_\sigma(s) \simeq s^\alpha$ *as* $s \to 0$, *for a constant* $\alpha > 1$ *independent of* σ.
If Γ *is a focus spiral associated with a system*

$$\begin{cases} \dot{x} &= -y + p(x,y) \\ \dot{y} &= x + q(x,y), \end{cases}$$

such that $p(x,y) = O(r^2)$ *and* $q(x,y) = O(r^2)$ *as* $r = \sqrt{x^2+y^2} \to 0$, *then*

$$\dim_B \Gamma = \begin{cases} 2 - \frac{2}{\alpha} & \text{for } \alpha > 2, \\ 1 & \text{for } 1 < \alpha \le 2. \end{cases}$$

Lyapunov coefficients

Definition (Lyapunov coefficients)

Consider a planar analytic system

$$\begin{cases} \dot{x} &= -y + p(x,y) \\ \dot{y} &= x + q(x,y), \end{cases} \tag{2}$$

with a weak focus at the origin, where $p(x,y)$ and $q(x,y)$ are analytic functions with all the terms of degree 2 or more. Then the Poincaré map for (2) near the focus for r sufficiently small can be written in the form

$$P(r) = r + \sum_{k=2}^{\infty} u_k r^k.$$

*The coefficient u_k in the above expansion is called k-*th Lyapunov coefficient *of the weak focus, $k \geq 2$. We denote the first nonzero Lyapunov coefficient by V_k.*

Box dimension of trajectory in the Hopf bifurcation

Theorem (D. Žubrinić, V. Županović (2008))

Assume that $p(x,y)$ and $q(x,y)$ are analytic functions with all the terms of degree 2 or more. Let Γ be a spiral trajectory near the origin of system

$$\begin{cases} \dot{x} &= ax - y + p(x,y) \\ \dot{y} &= x + ay + q(x,y), \end{cases} \tag{3}$$

where $a = 0$. If the first nonzero Lyapunov coefficient is V_3, then the Hopf bifurcation occurs at the origin of the system (3) at $a = 0$, and

$$\dim_B \Gamma = \frac{4}{3}.$$

Box dimension of trajectory of Liénard system

Theorem (D. Žubrinić, V. Županović (2008))

Let $a_{2k+1} \neq 0$ *in*

$$\begin{cases} \dot{x} & = & -y + \sum_{i=1}^{N} a_{2i} x^{2i} + \sum_{i=k}^{N} a_{2i+1} x^{2i+1} \\ \dot{y} & = & x, \end{cases} \tag{4}$$

that is, a_{2k+1} *is the first nonzero coefficient corresponding to an odd exponent of* x*. Then any spiral trajectory* Γ*, viewed near the origin, has box dimension equal to*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right). \tag{5}$$

Box dimension of trajectory of analytic systems

Theorem (D. Žubrinić, V. Županović (2008))

Let Γ *be a spiral trajectory near the origin of system*

$$\begin{cases} \dot{x} &= -y + p(x,y) \\ \dot{y} &= \quad x + q(x,y), \end{cases}$$

where $p(x,y)$ *and* $q(x,y)$ *are analytic functions with all the terms of degree* 2 *or more. If the first nonzero Lyapunov coefficient is* V_{2k+1}, *then*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right).$$

Generalized Bessel system

Planar system - nonautonomous

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\frac{\nu^2}{t^2} - 1)x - \frac{2-\mu}{t}y, \quad \nu \in \mathbf{R} \quad 0 < \mu < 2 \end{aligned}$$

Spatial system - autonomous using substitution $z = 1/t$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\nu^2 z^2 - 1)x - (2-\mu)yz, \quad \nu \in \mathbf{R} \quad 0 < \mu < 2 \\ \dot{z} &= -z^2 \end{aligned}$$

Main references

1. L. Korkut, D. Vlah, V. Županović, Fractal properties of Bessel functions, arXiv:1304.1762, submitted
2. M. Pašić, S. Tanaka, Fractal oscillations of self-adjoint and damped linear differential equations of second-order, Applied Mathematics and Computation, Vol. 218 (2011), 2281–2293.
3. D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, Bulletin des Sciences Mathématiques, Vol. 129 (2005), 457–485.
4. D. Žubrinić, V. Županović, Poincaré map in fractal analysis of spiral trajectories of planar vector fields, Bulletin of the Belgian Mathematical Society Simon Stevin, Vol. 15 (2008), 947–960.